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GEOMETRICAL SOLUTION OF SPHERICAL TRIANGLES

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1. Introduction. If three parts of a spherical triangle are given arcs, it at once follows from the formulas of spherical trigonometry that the other three parts can be constructed in a plane by ruler and compass. Actual constructions, moreover, have been given. Book [1] cited in the references at the end of this paper specifies, for example, certain plane triangles which contain the unknown parts or their functions as elements, so that the construction is effectively indicated, although the emphasis is upon the computation. The books [2], [3] employ descriptive geometry for constructing the unknown parts. A reading of Dutton's construction [3; 189, 190] led to the constructions to be given here.

With applications primarily in view the present paper (i) employs, except in §4, coördinates on the sphere, rather than spherical triangles; (ii) effectively replaces the three circles used by Dutton by a single circle; and (iii) gives a construction for finding the latitude and longitude of a point from which the altitudes of two stars have been simultaneously observed. The effect of (i) is to make the construction immediately and uniformly applicable in whatever quadrants the data of an astronomical problem lie; that of (ii) is to confine all points involved in the construction to the area of one circle and all measurement of arcs to the circumference of that circle. Among the advantages which accrue from (ii) is the possibility of designing an instrument to perform the operations mechanically. In §4 constructions for spherical triangles as such are deduced from the constructions developed in terms of coördinates.

It is to be noted that the treatment given here presupposes only certain almost intuitive results from spherical geometry and none from plane or spherical trigonometry. A class in solid geometry, for example, can easily be taught to find the distance between two points on the earth from their geographical coördinates. Furthermore, it is possible—and, if the construction is to be frequently used, highly desirable—to dispense with the oblique projection usually employed for representing a spherical triangle and to rely entirely on the representation in Figure 1, discussed in the next section. The reader versed in spherical trigonometry may, of course, prefer at the outset to follow the steps in Figure 4 on an auxiliary three-dimensional diagram.

It should further be remarked that the law of cosines, which effectively implies all the formulas of spherical trigonometry but is irrelevant to the present discussion, can be easily deduced in all generality from Figure 1. Conditions for the consistency of data can likewise be obtained as inequalities among arcs.

2. Representation on the plane. Figure 1 shows all that is essential for the construction. The positive sense for all angles is clockwise. For the point 0 on the earth's surface the circle shown represents the meridian on the celestial sphere and its opposite. This circle will be called the *fundamental circle.* Point 1 is (that is, represents) the north pole of the celestial sphere. Point 2 is the intersection of the meridian and the equator; it is 270° from 1. The zenith of 0 is 3.

The north and south points of the horizon are respectively 4, 5. The arc 2-3 is the latitude. A point on the celestial sphere is represented by its orthographic

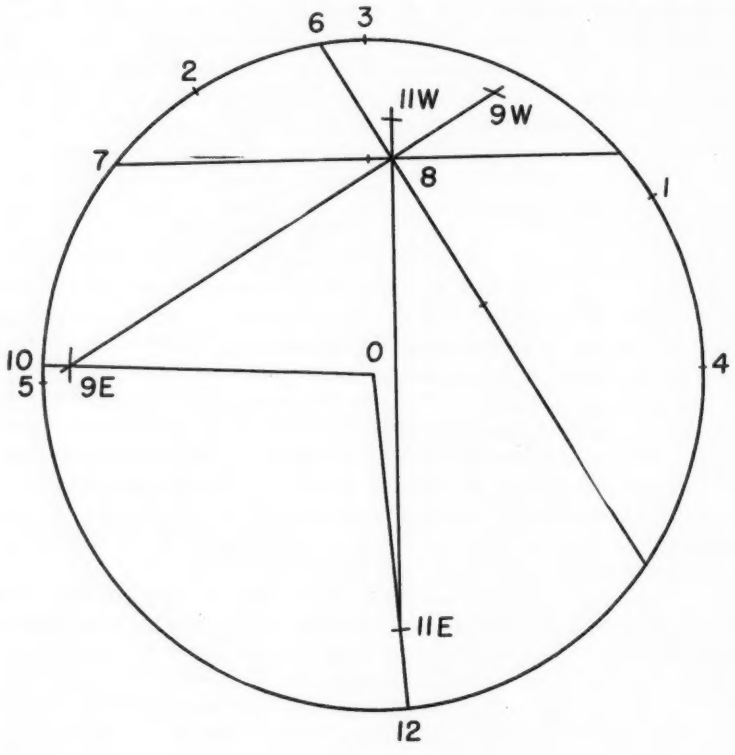

FIGURE 1

1	north pole	2–3	lat
2	equator	2–6	dec
3	zenith	2–10	LHA
0–9	radius diurnal circle	5–7	alt
0–11	radius altitude circle	4–12	az

projection 8 on the plane of the paper, the point of sight being to the east. To simplify language, we exclude from consideration the points on the fundamental circle; how to handle such points is obvious. Each point 8 interior to the circle represents exactly two points on the celestial sphere, one to the east (towards the reader) and one to the west. Arc 2-6 is the declination (north in the figure). The chord 8-6 is parallel to or coincides with 0-2 and is the projection of the diurnal circle. The representation of a given star (with fixed declination) moves with simple harmonic motion of period 24 sidereal hours along 8-6. Erect a perpendicular to 6-8 at 8. A circle with center at 0 and radius equal to that of

the diurnal circle (half the full chord 6-8) cuts the perpendicular in two points, one of them nearer 1 than the other. The one farther from 1 is numbered 9E and belongs to the point E in the east whose representation is 8. The one nearer to 1 is numbered 9W and belongs to the point W in the west. Let the radius 0-9E cut the fundamental circle in 10. Then the LHA of E is arc 2-10. The chord 8-7 parallel to or coinciding with 4-5 is the projection of the altitude circle. The altitude is arc 5-7. A circle with center 0 and radius equal to that of the altitude circle cuts the perpendicular to 7-8 at 8 in two points 11E, 11W, the one nearer to 3 being associated with the west. Let 0-11E cut the circle in 12. The azimuth of the point in the east represented by 8 is arc 4-12.

The proof of the foregoing is as follows. Let the diurnal circle be rotated about its projection 6-8 into the plane of the paper so that its western half is on the same side of 6-8 as is 1. Then translate the circle so that its center falls on 0, retaining only the points of intersection 9E, 9W with the perpendicular. The altitude circle is treated similarly.

3. Application of the representation. Although the construction just given is truly ruler-and-compass, it is convenient in applications to employ a transparent circular protractor which has the same size as the fundamental circle and is graduated clockwise from 0° to 360°. This permits the data and answers to be expressed numerically.

For the present suppose that the latitude and longitude are given. The fundamental construction problem is this: given one of the sets of coördinates

(dec, GHA), (alt, az), (dec, alt)

for a point on the celestial sphere, find either or both of the others. The pair (dec, alt) must, of course, be accompanied by a designation east or west in order that the point represented be unique.

Any one of the phases of the problem is immediately solved by reference to Figure 1. If GHA is to be read or set, the protractor is placed so that the reading at 2 is the longitude (the value between 0° and 360°) of the point whose zenith is 3.

Star charts for a given time and place can readily be constructed from the Nautical Almanac. The representative points 8 for the stars in the west can be constructed exactly as described. It is convenient to have those in the east on a separate chart and, since they are to be viewed from the west, to reflect the whole of Figure 1 in a vertical line by moving 1 to the left and turning the protractor over so that angles are measured counterclockwise. To construct subsequently a chart for the stars in the south, construct the point corresponding to 11 by measuring the azimuth from the lowest point of the circle and then project on the horizontal chord as before. In general, if the azimuth of the observer (that is, the azimuth of the point of sight for projection) is B, shift clockwise through $90° - B$ the point 4 from which the azimuth is measured in Figure 1; that is, rotate each horizontal projection 11 through B-90° into 11′ and determine the new representation 8′ by projecting 11′ orthogonally upon the unaltered trace 7-8 of the altitude circle.

4. Solution of spherical triangles. When a spherical triangle is proposed for solution, the problem will be called *lateral* or *angular* according as at least two sides or at least two angles are given. If a problem is angular, the corresponding problem for the polar triangle is lateral. Hence we limit ourselves to the discussion of lateral problems.

Given (a, b, c) to find A we may take $a=90°-\text{alt}$, $b=90°-\text{dec}$, $c=90°-\text{lat}$ and determine A as the GHA on the assumption that the longitude is 0 and that the star is in the west. It is convenient, however, to rearrange matters for direct reading with a semicircular protractor as has been done in Figure 2. Signs being ignored, a can be measured from 3 and b, c from 1. It is also convenient to measure A from 1: this can be done by using a perpendicular to 0-9 rather than 0-9

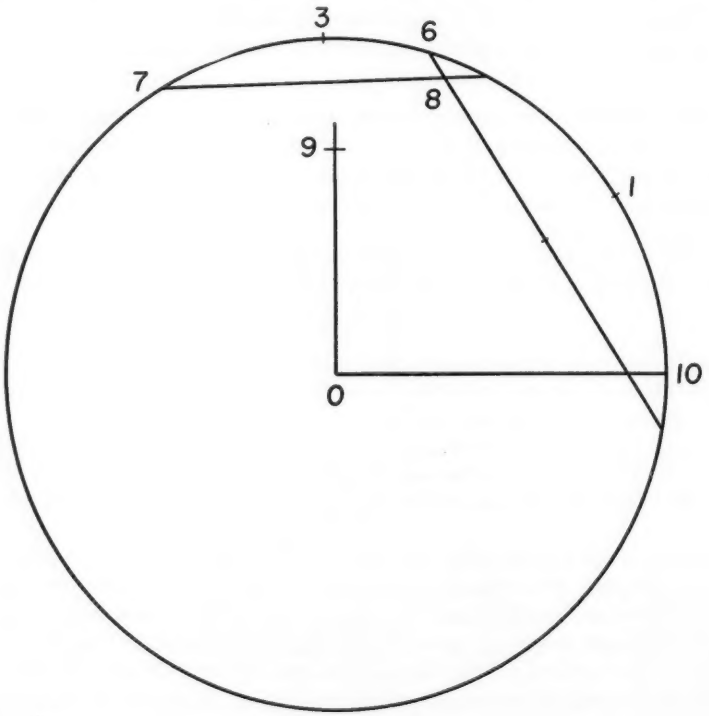

FIGURE 2

3–7 = a, 1–6 = b, 1–3 = c, 1–10 = A.

itself. It is not necessary to invoke the rule for choosing between 9E and 9W: if 9E is used, a numerically equal negative A is found. The reason behind this is that the construction is analytically equivalent to the equation of the law of

cosines in which A appears only through its cosine. Figure 2 is constructed for $a=30°$, $b=42°$, $c=60°$, whence $A=32°$, roughly.

Given (A, b, c) to find a, we use exactly the same figure, drawing 7-8 last.

If (a, b, A) are given, the construction is illustrated in Figure 3, in which it is found convenient to place the pole 1 at the top since the zenith is unknown. Points 6, 8, 9 are found much as before, and the rest of the problem is essentially to give chord 7-8 the proper direction. Point 13 is found by striking from 9E, 9W arcs of radius equal to half the chord of an arc of length $2a$ in the fun-

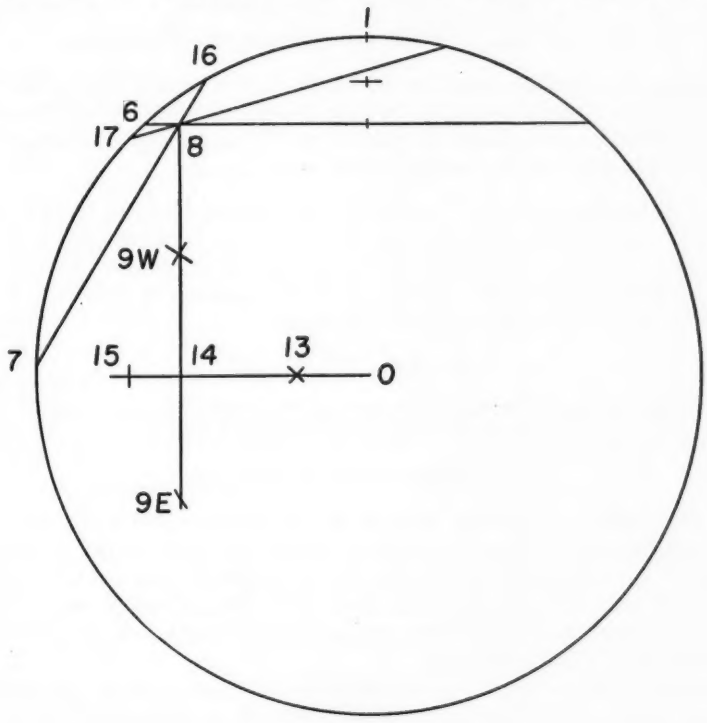

FIGURE 3

damental circle: the radius was found in Figure 3 by marking arc a on either side of 1. Point 15 is obtained by striking an arc of the radius 13-9W with 13 as center. Points 16, 17 are found by striking arcs of radius 14-15 with 8 as center. The chord or chords 7-8 are obtained by joining 8-16 and 8-17. Since c must be positive, point 6 must lie in the arc less than 180° subtended by any chord which is to give a solution.

The situations which can arise with arbitrary data are quite varied. If each of the given parts is between 0° and 180°, it is convenient to divide the situa-

tions into five mutually exclusive and exhaustive types illustrated by the following numerical data:

	a	b	A	
(4.1)	30°	60°	45°	No solution.
(4.2)	30°	45°	45°	One solution in which $B = 90°$.
(4.3)	30°	20°	45°	One solution in which $B \neq 90°$.
(4.4)	30°	40°	45°	Two solutions.
(4.5)	90°	90°	90°	Infinitely many solutions.

In passing, it should be remarked that type (4.5) is usually overlooked in the analysis of the ambiguous case given in textbooks, possibly because the reasoning in them follows too closely the pattern for the plane, where a circle of radius a cannot coincide with the geodesic which supports side c.

5. The location problem. The so-called astronomical triangle has for vertices the three points North Pole, Zenith, Star. Each of these vertices is the pole of a system of polar coördinates on the sphere. Among the coördinates of the vertices themselves in these systems, three are replaced by their complements and with two of the angles are given the names

(5.1) (lat, dec, LHA, alt, az).

In addition, the polar system with pole at the north pole is connected with a fixed polar system, in which the angular coördinate is GHA, by the equation

(5.2) $$\text{GHA} = \text{LHA} + \text{long}.$$

It is necessary in problems such as the one we are about to discuss to consider simultaneously several triangles of points analogous to the astronomical triangle. We shall employ the same names as appear in (5.1) for the analogous coördinates and to avoid confusion shall write $(\text{alt, az})_{ABC}$ to mean "the coördinates analogous to alt, az when the north pole, zenith and star are replaced by points A, B, C, respectively."

Let now P be the north pole, Z the zenith of a fixed point on the earth, not at either pole, and S, T two stars whose GHA's do not differ by 0° or 180° and neither of which coincides with P or Z. Suppose further that the notation is adjusted so that star S is to the eastward in the sense that

(5.3) $$\text{GHA}T - \text{GHA}S \equiv D \pmod{360°},$$

where $0° < D < 180°$. Set

$$(\text{long, lat, dec, GHA, alt})_{PZS} = (x, y, d_1, G_1, a_1),$$

$$(\text{dec, GHA, alt})_{PZT} = (d_2, G_2, a_2).$$

We suppose that the observer measures a_1, a_2 simultaneously at GCT t, that he notes whether the orientation STZ is positive (clockwise as viewed

from outside the celestial sphere) or negative and whether S is to the east or west of the meridian, and that he determines (d_1, G_1), (d_2, G_2) for time t from the Nautical Almanac. Our problem is then to find (x, y) by construction.

The construction consists of three changes of coördinates of the nature described in §3. They are described in tabular form below.

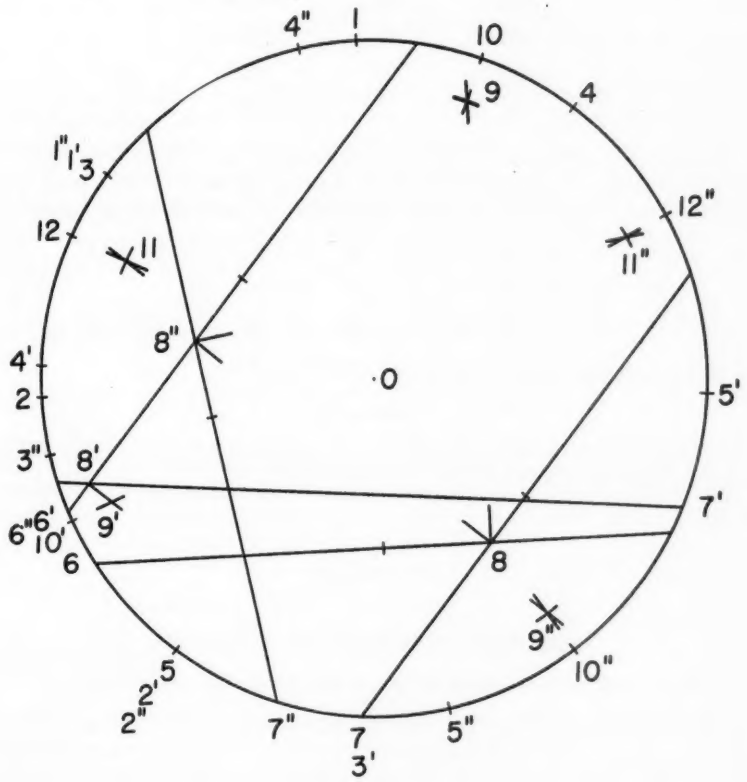

FIGURE 4

	Given	Find
(i)	$(\text{lat, dec, LHA})_{PST} = (d_1, d_2, D)$	$(\text{alt, az})_{PST} = (p, q)$
(ii)	$(\text{lat, dec, alt})_{STZ} = (p, a_1, a_2)$	$(\text{LHA})_{STZ} = u.$
(iii)	$(\text{lat, dec, LHA})_{SPZ} = (d_1, a_1, q + u)$	$(\text{alt, az})_{SPZ} = (y, v).$

The longitude is given by

$$x = \text{GHA}S + v.$$

The data for constructing Figure 4 and the coördinates determined for Z are shown in the table below.

	PST	*STZ*	*SPZ*	
	+	+	−	
	dec	alt	GHA	*D*
S	40°	30°	236°	
T	−30°	20°	348°	112°
Z	30°		310°	

The notation in Figure 4 is the same as that in Figure 1, the two extra stages being distinguished by accents. Portions of some of the lines have been omitted for convenience. The order in which the points are obtained is as follows:

1 2 3 4 5 6 10 9 8 7 11 12

1′ 2′ 3′ 4′ 5′ 6′ 7′ 8′ 9′ 10′

1″ 2″ 3″ 4″ 5″ 6″ 10″ 9″ 8″ 7″ 11″ 12″

The position of the unknown point Z is (50°E, 30°N).

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1. J. D. H. Donnay, Spherical Trigonometry after the Cesàro method, New York, 1945.
2. J. F. Dowsett, Advanced Constructive Geometry, London, 1927.
3. B. Dutton, Navigation and Nautical Astronomy, 7th edition, United States Naval Institute, Annapolis, Md., 1942.

ON THE THEOREM OF FRULLANI

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1. Introduction. Among the papers left by the late Professor Harry Bateman, there is much material on definite integrals which is now being used for the construction of a table of integrals.* In this material, a little known theorem of an Italian mathematician of the first half of the nineteenth century, the theorem of Frullani [1] plays quite an extensive role. This theorem makes possible in many cases the evaluation of an improper integral of the type

$$\int_0^\infty [f(ax) - f(bx)] \frac{dx}{x} \tag{1}$$

or of integrals of more general types, even when the two integrals of which (1) is the difference, do not exist separately.

* This work is sponsored by the Office of Naval Research.

Frullani's formula,

$$\int_0^\infty [f(ax) - f(bx)]\frac{dx}{x} = f(0)\log\frac{b}{a}, \tag{2}$$

and the more general result,

$$\int_0^\infty [f(ax) - f(bx)]\frac{dx}{x} = [f(0) - f(\infty)]\log\frac{b}{a}, \tag{3}$$

have been investigated recently [2] with a view of obtaining the least restrictive conditions on f which will ensure the validity of the previous results.

The purpose of this paper is something else. Precisely, I generalize the formulas (2) and (3) in two directions, first, by using two, in some degree arbitrary, functions $p_1(x)$ and $p_2(x)$ in place of the linear functions ax and bx, and, second, by allowing the constants a and b (which are real in Frullani's work) to assume complex values. The list of references collected by Bateman was very helpful to me.

2. The generalization. Let us consider the integral,

$$I_{\delta\omega} = \int_\delta^\omega \left\{f[p_1(x)]\,\frac{p_1'(x)}{p_1(x)} - f[p_2(x)]\,\frac{p_2'(x)}{p_2(x)}\right\}dx \qquad (0 < \delta < \omega), \tag{4}$$

where for the sake of simplicity we shall impose the unnecessarily restrictive conditions that $f(x)$ is continuous, and that $p_1(x)$ and $p_2(x)$ are positive and continuously differentiable, for $x > 0$.

With $p_1(x) = y_1$, $p_2(x) = y_2$,

$$I_{\delta\omega} = \int_{p_1(\delta)}^{p_1(\omega)} f(y_1)\,\frac{dy_1}{y_1} - \int_{p_2(\delta)}^{p_2(\omega)} f(y_2)\,\frac{dy_2}{y_2} = \int_{p_1(\delta)}^{p_2(\delta)} - \int_{p_1(\omega)}^{p_2(\omega)} f(y)\,\frac{dy}{y} \tag{5}$$

and hence

$$\int_0^\infty \left\{f[p_1(x)]\,\frac{p_1'(x)}{p_1(x)} - f[p_2(x)]\,\frac{p_2'(x)}{p_2(x)}\right\}dx = \lim_{\delta\to 0,\,\omega\to\infty} I_{\delta\omega} = L_0 - L_\infty, \tag{6}$$

where

$$L_0 = \lim_{\delta\to 0}\int_{p_1(\delta)}^{p_2(\delta)} f(y)\,\frac{dy}{y}, \qquad L_\infty = \lim_{\omega\to\infty}\int_{p_1(\omega)}^{p_2(\omega)} f(y)\,\frac{dy}{y}. \tag{7}$$

Thus we see that the integral (6) can be evaluated whenever we are able to evaluate both limits (7).

Now, by the first mean value theorem of integral calculus,

$$\int_{p_1(\xi)}^{p_2(\xi)} f(y)\,\frac{dy}{y} = f(p)\log\frac{p_2(\xi)}{p_1(\xi)}$$

where p is a suitable value between $p_1(\xi)$ and $p_2(\xi)$, and so whenever

$$\lim_{x\to+0} f(x) = f(0), \qquad \lim_{x\to+0} \frac{p_2(x)}{p_1(x)} = \lambda_0 \neq 0;$$

$$\lim_{x\to+\infty} f(x) = f(\infty), \qquad \lim_{x\to+\infty} \frac{p_2(x)}{p_1(x)} = \lambda_1 \neq 0 \tag{8}$$

exist, the two limits (7) and (8) also exist, and

$$L_0 = f(0) \log \lambda_0, \qquad L_\infty = f(\infty) \log \lambda_1.$$

Hence in this case

$$\int_0^\infty \left\{ f[p_1(x)] \frac{p_1'(x)}{p_1(x)} - f[p_2(x)] \frac{p_2'(x)}{p_2(x)} \right\} dx = f(0) \log \lambda_0 - f(\infty) \log \lambda_1. \tag{9}$$

If $p_1(x) = ax$, $p_2(x) = bx$ and $a, b > 0$, this reduces to (3). A formula equivalent to the particular case $p_1(x) = x$ of (9) was given by Lerch [3].

If only the first two limits in (8) exist, and in addition

$$\lim_{x\to+\infty} p_1(x) = \lim_{x\to+\infty} p_2(x) = +\infty \tag{10}$$

and the infinite integral

$$\int_h^\infty f(y) \frac{dy}{y}, \qquad (h > 0), \tag{11}$$

exists, then clearly $L_\infty = 0$, and we have

$$\int_0^\infty \left\{ f[p_1(x)] \frac{p_1'(x)}{p_1(x)} - f[p_2(x)] \frac{p_2'(x)}{p_2(x)} \right\} dx = f(0) \log \lambda_0, \tag{12}$$

which reduces to (2) when $p_1(x) = ax$, $p_2(x) = bx$ and $a, b > 0$.

It is worth noting that there is a relation between the present matter and the theory of the Cauchy principal value of an integral. In (5), we change y_2 into $-y_2$, and define $f(y)$ for negative y by the relation $f(-y) = f(y)$. Then

$$I_{0\omega} = \lim_{\delta\to 0} \int_{-p_2(\omega)}^{-p_2(\delta)} f(y) \frac{dy}{y} + \int_{p_1(\delta)}^{p_1(\omega)} f(y) \frac{dy}{y}$$

so that under the assumptions

$$\lim_{x\to+0} p_1(x) = \lim_{x\to+0} p_2(x) = 0, \quad \lim_{x\to+0} \frac{p_2(x)}{p_1(x)} = \lambda_0 \neq 0, \quad \lim_{x\to+0} f(x) = f(0) \tag{13}$$

it follows that

$$I_{0\omega} = \int_{-p_2(\omega)}^{*\,p_1(\omega)} f(y)\frac{dy}{y} + f(0)\log\lambda_0, \tag{14}$$

where the asterisk denotes the principal value of the integral. A similar formula is valid for $I_{\delta\omega}$ and, consequently, even for $I_{\delta\omega}$.

3. The case of a periodic function. Another case of interest arises when only the first two limits in (8) exist, and instead of the convergence of (11) we assume that $f(y)$ is an integrable periodic function with period p.

As is well known, in this case the corresponding indefinite integral

$$F(x) = \int_0^x f(y)dy$$

is the sum of a (continuous) periodic function $\phi(x)$ and the linear function Mx, where M is the mean value of $f(x)$ over the period, *i.e.*,

$$M = \frac{1}{p}\int_0^p f(y)dy.$$

Next, if we consider another function

$$\Phi(x) = \int_h^x f(y)\frac{dy}{y}, \qquad (h > 0),$$

we have, by integration by parts,

$$\begin{aligned}\Phi(x) &= \left[\frac{1}{y}F(y)\right]_h^x + \int_h^x F(y)\frac{dy}{y^2} \\ &= \left[M + \frac{1}{y}\phi(y)\right]_h^x + M\int_h^x y\frac{dy}{y^2} + \int_h^x \phi(y)\frac{dy}{y^2} \\ &= \frac{1}{x}\phi(x) - \frac{1}{h}\phi(h) + M\log x - M\log h \\ &\quad + \int_h^\infty \phi(y)\frac{dy}{y^2} - \int_x^\infty \phi(y)\frac{dy}{y^2}.\end{aligned}$$

Consequently, if the constant H is defined as

$$H = -\frac{1}{h}\phi(h) - M\log h + \int_h^\infty \phi(y)\frac{dy}{y^2}$$

and we observe that, by means of the substitution $y = xt$,

$$\int_x^\infty \phi(y)\frac{dy}{y^2} = \frac{1}{x}\int_1^\infty \phi(xt)\frac{dt}{t^2} = O\left(\frac{1}{x}\right), \tag{15}$$

we may write

$$\Phi(x) = M \log x + H + O(1/x).$$

In view of this formula (which is interesting in itself) we have

$$\int_{p_1(\omega)}^{p_2(\omega)} f(y)\frac{dy}{y} = \Phi[p_2(\omega)] - \Phi[p_1(\omega)]$$

$$= M \log \frac{p_2(\omega)}{p_1(\omega)} + O\left(\frac{1}{p_1(\omega)}\right) + O\left(\frac{1}{p_2(\omega)}\right).$$

Therefore, if in addition to conditions (10) we suppose that

$$\lim_{x\to+\infty} \frac{p_2(x)}{p_1(x)} = \lambda_1 \neq 0, \tag{16}$$

we can write the desired formula

$$\int_0^\infty \left\{ f[p_1(x)]\frac{p_1'(x)}{p_1(x)} - f[p_2(x)]\frac{p_2'(x)}{p_2(x)} \right\} dx = f(0)\log\lambda_0 - M\log\lambda_1. \tag{17}$$

In the particular case $p_1(x)=ax$, $p_2(x)=bx$, a formula similar to (17) was found by Schlömilch [4] but Schlömilch's method (expansion in a Fourier series) introduces unnecessary restrictions on the function f.

With regard to generality, it should be noted further that in the preceding proof the only essential property of $\phi(x)$ is that the integral (15) vanishes as $x\to+\infty$. Consequently the proof is not only valid for periodic functions but for all integrable functions f for which a constant M exists such that we can write

$$F(x) = \int_0^x f(y)dy = Mx + o(x),$$

i.e., for all the functions f with a finite mean value on the whole positive half axis:

$$\lim_{x\to+\infty} \frac{1}{x}\int_0^x f(y)dy = M.$$

4. Extension into the complex field. For sake of simplicity, in this last section we shall limit ourselves to the case $p_1(x)=ax$, $p_2(x)=bx$. It is well known, that by means of the substitutions,

$$x = e^\xi, \qquad a = e^\alpha, \qquad b = e^\beta, \qquad f(e^\xi) = F(\xi), \tag{18}$$

the equation (3) can be written in the form,

$$\int_{-\infty}^{\infty} [F(\xi+\alpha) - F(\xi+\beta)]d\xi = (\beta-\alpha)[F(-\infty) - F(\infty)], \tag{19}$$

where $F(-\infty)$ and $F(\infty)$ represent the limit values of $F(\xi)$ as $\xi\to\mp\infty$, and α

and β are two arbitrary (positive, negative or zero) real constants.

In this form, if F is an analytic function of ξ, Frullani's theorem can easily be extended to the complex field, *i.e.*, to the case of two arbitrary complex numbers α and β, where it will be assumed that $\Im\alpha \leqq \Im\beta$. Moreover we shall suppose: (i) that in the strip $\Im\alpha \leqq \Im\xi \leqq \Im\beta$ there lies a finite number of singular points of $F(\xi)$, and none are on the boundary of the strip; (ii) that, as $\Re\xi \to \pm\infty$, $F(\xi)$ approaches the limit values $F(\infty)$ and $F(-\infty)$, respectively, uniformly in the strip.

Under this hypothesis we have

$$\int_{-\infty}^{\infty} [F(\xi+\alpha) - F(\xi+\beta)]\,d\xi = (\beta-\alpha)[F(-\infty) - F(\infty)] + 2\pi i S, \tag{20}$$

where the path of integration is the real axis and S is the sum of the residues of F in the strip (without considering the point at infinity).

This formula is easily demonstrated by applying the theorem of residues to the integral of $F(\xi)$ taken around the parallelogram with vertices $\alpha-\omega_1$, $\alpha+\omega_2$, $\beta+\omega_2$, $\beta-\omega_1$, (ω_1, ω_2 positive) and passing to the limit as ω_1 and ω_2 approach infinity. This demonstration was given by G. H. Hardy and H. W. Carjel in a solution to a problem proposed by E. E. Elliot in the *Educational Times* [5].

However, it does not seem to have been observed that even the Cauchy-Elliot equation (3) can be extended to the complex field by a transformation of equation (20) by means of the inverses of the substitutions (18). But here we must bear in mind the fact that these inverse substitutions are generally multiple-valued. They become single-valued, however, if we assume that the imaginary parts of ξ, α, β are in $(-\pi, \pi)$, and this does not impose restrictions on the three complex numbers x, a, b, other than that they are different from zero. Moreover, it should be noted that a line parallel to the real axis in the ξ-plane corresponds to a half-line issuing from the origin in the x-plane.

Thus we obtain the formula,

$$\int_0^{\infty} [f(ax) - f(bx)]\frac{dx}{x} = (\log b - \log a)[f(0) - f(\infty)] + 2\pi i S^*, \tag{21}$$

where:

(i) the imaginary parts of both logarithms lie in the range $(-\pi, \pi)$;

(ii) if γ is the sector $\arg a \leqq \arg x \leqq \arg b$, then S^* represents the sum of the residues of $f(x)$ inside γ, each divided by the affix of the corresponding singular point;

(iii) $f(0)$ and $f(\infty)$ represent respectively the limit values of $f(x)$ as $|x| \to 0$ and $|x| \to \infty$ in γ with the condition that $f(x)$ approaches these limits uniformly with respect to $\arg x$.

For the proof, it is sufficient to observe that if the analytic function $F(\xi)$ has a singular point with residue R at $\xi = \xi_0$, the function $f(x) = F(\log x)$ has a

singular point with residue $x_0 R$ at $x = x_0 = e^{\xi_0}$. In fact, we have

$$\frac{R}{\xi - \xi_0} = \frac{R}{\log x - \log x_0} = \frac{R}{\log\left(1 + \dfrac{x - x_0}{x_0}\right)} = \frac{R}{\dfrac{x - x_0}{x_0} - \dfrac{1}{2}\left(\dfrac{x - x_0}{x_0}\right)^2 + \cdots}$$

$$= \frac{Rx_0}{x - x_0} + \cdots .$$

Finally, it should be noted that if some simple poles of $f(x)$ lie on the boundary of γ, formula (21) remains valid providing the integral on the left is interpreted as a Cauchy principal integral and the residues at the corresponding poles are halved in the sum S^*.

For example, in the case

$$f(x) = e^{-x}, \qquad a = a_1 + ia_2, \qquad b = b_1 + ib_2, \qquad (a_1 > 0, b_1 > 0)$$

we have

$$f(0) = 1, \qquad f(\infty) = 0, \qquad S^* = 0,$$

and (21) gives us, by separating the real and imaginary parts, the interesting formulas,

$$(22') \qquad \int_0^\infty \left[e^{-a_1 x} \cos(a_2 x) - e^{-b_1 x} \cos(b_2 x)\right] \frac{dx}{x} = \frac{1}{2} \log \frac{b_1^2 + b_2^2}{a_1^2 + a_2^2},$$

$$(22'') \qquad \int_0^\infty \left[e^{-a_1 x} \sin(a_2 x) - e^{-b_1 x} \sin(b_2 x)\right] \frac{dx}{x} = \arctan \frac{a_2}{a_1} - \arctan \frac{b_2}{b_1},$$

where both arctangents are taken between $-\pi/2$ and $\pi/2$ (since the real parts of a and b are positive). This is a generalization of a known result of Poisson [6].

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2. Iyengar, J. Indian Math. Soc., vol. 4, 1940, 145–150; Proc. Cambridge Phil. Soc., vol. 37, 1941, 9–18. More complete results in this direction were obtained by Professor A. Ostrowski of Basel who reported on his (as yet unpublished) work to the Unione Matem. Italiana (Pisa, September 1948) and to the University of California (Los Angeles, May, 1949).

3. M. Lerch, Abhand. Akad. d. Wiss. Prag vol. 1^3, 1891 pp. 123–131. See also G. H. Hardy, Messenger of Math. (2) vol. 34, 1905 pp., 11–18 and 102.

4. O. Schlömilch, Archiv. Math. u. Physik (Grunert) (1) 5 (1866) pp. 152–155.

5. Educational Times v. 72 (1900) pp. 89–90, Question 14164.

6. S. D. Poisson, Jour. École Polyt. vol. 9, 1813, p. 215 (see pp. 220–221). The formula of Poisson is (22'') when $b_2 = 0$.

MATHEMATICS FOR THE MILLION, OR FOR THE FEW?*

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It is now thirteen years since the appearance of a much-debated volume entitled "Mathematics for the Million." That title was decidedly misleading. In reality, the book presupposed a mathematical maturity ordinarily achieved only by "the few." However, it served to focus attention on the real question at issue; namely, what kind of mathematical education should the schools provide not merely for a gifted minority, but also for the average citizen?

Ever since the organization of the International Commission on the Teaching of Mathematics, at the Fourth International Congress of Mathematicians, in Rome, Italy, in 1908, published reports on the mathematical curricula of the participating nations have gradually become available. In the United States, summaries of these documents have appeared in such publications as the National Report of 1923 and the Fourth Yearbook of the National Council of Teachers of Mathematics. They made it quite clear that in the majority of the reporting nations the schools provided one type of education for the masses and a radically different kind of education for the "classes." That kind of program is possible only in a dual system of education consisting of elementary schools for the many and secondary schools for the few.

However, in the United States that arrangement has long been considered undemocratic. Instead, the prevailing policy is that of admitting "all the children of all the people" to every kind of public school. Accordingly, our schools have become a sort of educational laboratory on a huge scale. And it may well be that both our successes and our failures in the field of mass education hold valuable lessons for the other nations. In the light of this background, let us briefly examine what has been done here, or is being proposed, by way of answering the question embodied in the title of this paper.

Three principal approaches have been tried or suggested in American secondary schools in their attempts to provide functioning types of mathematical training for all American youth.

The first is anchored on the popular doctrine that "life situations" should be the primary basis of all curricula. But since mathematics is a system of ideas and processes, whereas life situations are incurably unsystematic, this approach has failed completely wherever it has been tried. It has always resulted merely in a sort of chaotic "mathematics without mathematics" and it ignored fundamental aspects of the problem we are considering.

The second approach hopes to find dependable answers in the recommendations of authoritative committees and in the techniques of curriculum workshops and laboratories. However, the thousands of mathematical syllabi now crowding the shelves of our curriculum morgues have merely dramatized a hopeless confusion of objectives. And even the reports of national committees

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have regularly been attacked by leading educators as mere reflections of unsupported private opinion.

There remains a third approach, often explored partially, but never with anything like scientific completeness or thoroughness. It is that of making a really dependable, full-length study of the role of mathematics in the modern world, from both a practical and a cultural standpoint.

To accomplish that, we should have to look carefully at many aspects of the question. How and where do we need mathematics in our homes, our vocations and professions, as wage earners, as citizens, as readers, as students, and so on? We should have to find out how mathematics is related to the industrial, commercial, and engineering enterprises of the nation. And in what way do mathematical modes of thinking, attitudes, and types of appreciation permeate the very warp and woof of modern civilization? To obtain the information we need, it would be necessary to analyze the work of many groups of people, such as businessmen, builders, contractors, mechanics, machinists, designers, draftsmen, economists, scientists, statisticians, accountants, navigators, and pilots. When such studies become available, we are sure to discover the utterly untenable character of the idea that only "the few" need genuine mathematical training. On the contrary, it will be found that mathematics is a subject of global significance, and that a citizen of the modern world simply cannot afford to be ignorant of a basic knowledge of mathematics, which must surely involve a little more than grocery store arithmetic and a few rules of mensuration.

It is believed that only on the basis of such objective findings will it be possible to obtain a convincing and enduring foundation for the mathematical curricula of the modern school. It will then be our major task to build a two-track program, or even a three-track program, for the divergent needs of those seeking an immediate life preparation and of those who aim at foundational studies in our colleges and universities.

This brief report is submitted in the ardent hope that it may enlist the active concern of another International Congress of Mathematicians pertaining to this "unfinished business," namely, our common problem of providing more adequate curricula both for the many and the few. Is it too much to expect that a work which was begun so auspiciously in 1908, at the instance of our own David Eugene Smith, be resumed and eventually brought to more complete consummation? It is therefore urged that Section VII of this Congress transmit to the International Mathematical Union an expression of its profound interest in this problem, together with the definite suggestion that proper steps be taken forthwith to initiate its solution.

AN ACKNOWLEDGMENT

Theorem 1 of my paper *Some applications of matrices in the theory of equations* (this MONTHLY 57 (1950), 156) is equivalent to an earlier result of W. V. Parker and M. M. Flood (this MONTHLY 42 (1935), 164 and 43 (1936), 562).

C. C. MACDUFFEE

ART AND MATHEMATICS: A BRIEF GUIDE TO SOURCE MATERIALS

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Sometime ago the writer asked a person well-known in the field of modern art for some material which would show the relation of geometry to art. Somewhat to his amazement came the crisp reply that there is no relation between art and geometry. Upon delving into the literature, it became apparent that there are two schools of thought about the matter, and that considerable expository material, in scattered form and not always readily accessible, nevertheless does exist.

Not only for the immediate question of the relation between art and mathematics, but also for the education of teachers of mathematics, the matter would seem to deserve consideration. It is generally conceded that the well-qualified teacher must be competent in each of three areas: (1) he must understand the ways of youth, and the nature of learning; (2) he must be well versed in mathematics; and (3) he must have attained sound bearings in some of the other domains of human achievement—at least in the physical sciences, the social disciplines, the crafts and technologies, and the fine arts. If this thesis is accepted, then clearly what is demanded is not merely a professionally trained person, but a truly cultured individual.

To be sure, much of a teacher's experience and attainment in this third category are secured in the course of time, as his interest, opportunities and resources are likely to dictate. Frequently, however, the resources are not as convenient as could be desired; the material is not always at hand, or it is scattered, or adequate guides to source materials are lacking. This is perhaps more true of material dealing with mathematics and art than of material dealing with mathematics and science, or mathematics and technology. Hence it is to be hoped that the following list of reference materials on art and mathematics, despite its incompleteness and other shortcomings, may be helpful to mathematics teachers in service, to prospective teachers of mathematics (and of art), to students of the subject, and to lay readers as well.

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MATHEMATICAL NOTES

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ON THE QUADRATIC RECIPROCITY LAW

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1. Introduction. Kronecker [2] obtained the quadratic reciprocity law for rational primes p, q by considering the decomposition of the prime p into prime ideal factors in the quadratic extension $R(\eta)$ of the rational number field R generated by a root η of the period equation $x^2+x+[1-(-1|q)q]/4=0$. However, Kummer [see **5**, p. 78] had shown that the decomposition of p in $R(\eta)$ parallels in every respect the decomposition of $x^2+x+[1-(-1|q)q]/4$ modulo p; and Gauss [**1**], Lebesgue [**3**], and Pellet [**4**] obtained the reciprocity law by using the theory of higher congruences to discuss directly the decomposition of $x^2+x+[1-(-1|q)q]/4$ modulo p. The theory of Galois (finite) fields as distilled from the older theory of higher congruences is discussed in many recent texts on modern algebra and number theory; and though various well-known proofs of the reciprocity law, including that of Kronecker mentioned above, appear in these texts, none are based on this theory. A proof based on Galois field theory is both simple and direct, and we give such a proof here, borrowing a little from each of the above mentioned proofs of Gauss, Lebesgue, and Pellet.

2. Proof. Let p and q be distinct primes, q odd and $q=4k\pm1$. If p belongs to the exponent e modulo q, the Galois field $GF[p^e]$ contains a primitive qth root of unity ρ since the multiplication group of $GF[p^e]$ is cyclic of order p^e-1. The $(q-1)/2$-nomial periods of ρ are $\eta=\sum_a\rho^a$, $\eta'=\sum_b\rho^b$, where a ranges over the quadratic residues and b ranges over the quadratic non-residues modulo q.

Now η and η' are roots of a quadratic equation in $GF[p]$; to obtain the coefficients of this period equation, we need to find $\eta+\eta'$ and $\eta\eta'$. The roots of $x^q-1=0$ are 1 and the primitive qth roots of unity. Hence $\eta+\eta'=-1$. To find $\eta\eta'=\sum_{a,b}\rho^{a+b}$, we need to determine the number of distinct representations (mod q) of each of the residues $r=0, 1, \cdots, q-1$ in the form $a+b$. We note that the number of distinct representations of a given non-zero residue r in

the form $a+b$ is equal to the number of distinct representations of 1 in this form. The residue 0 admits of no representation in the form $a+b$ if $(-1|q)=1$, and exactly $(q-1)/2$ distinct representations if $(-1|q)=-1$.

Hence if $(-1|q)=1$, the number of distinct representations of each non-zero residue r in the form $a+b$ is exactly $(q-1)/4$ and we have

$$\eta\eta' = \frac{q-1}{4}\cdot\sum_{(r\neq 0)}\rho^r = \frac{1-q}{4}.$$

But if $(-1|q)=-1$, the number of distinct representations of each non-zero residue r in the form $a+b$ is exactly $(q-3)/4$ and we have

$$\eta\eta' = \frac{q-1}{2}+\frac{q-3}{4}\cdot\sum_{(r\neq 0)}\rho^r = \frac{1+q}{4}.$$

Therefore the quadratic equation in $GF[p]$ having η and η' as roots is $x^2+x-(-1|q)k=0$.

Incidental to the above argument is the fact that $(-1|q)=1$ implies $q\equiv 1 \pmod 4$, and $(-1|q)=-1$ implies $q\equiv 3 \pmod 4$. Hence this gives

$$(-1\,|\,q) = (-1)^{(q-1)/2},$$

the first supplement to the reciprocity law.

We observe that $\eta\neq\eta'$, since $\eta=\eta'$ with $\eta+\eta'=-1$ and $\eta\eta'=-(-1|q)k$ imply $4(-1|q)k=-1$ in $GF[p]$, and this in turn implies $q\equiv 0$ modulo p, a contradiction.

If now $(p|q)=1$, then

$$\eta^p = \left[\sum_a \rho^a\right]^p = \sum_a \rho^{pa} = \sum_a \rho^a = \eta,$$

and η is an element of $GF[p]$ since the roots of $x^p-x=0$ are the p elements of $GF[p]$. But if $(p|q)=-1$, then

$$\eta^p = \left[\sum_a \rho^a\right]^p = \sum_a \rho^{pa} = \sum_b \rho^b = \eta',$$

and η is not an element of $GF[p]$ since $\eta\neq\eta'$ and hence not a root of $x^p-x=0$. Thus $x^2+x-(-1|q)k$ splits into linear factors in $GF[p]$ if and only if $(p|q)=1$.

On the other hand, the discriminant of $x^2+x-(-1|q)k$ is $q(-1|q)$, and hence if p is odd, $x^2+x-(-1|q)k$ splits into linear factors in $GF[p]$ if and only if $(q(-1|q)|p)=1$. This with the above result gives

$$(p\,|\,q) = (q(-1)^{(q-1)/2}\,|\,p),$$

the Gaussian form of the reciprocity law.

Finally, if $p=2$, $x^2+x-(-1|q)k$ splits into linear factors in $GF[p]$ if and only if $k\equiv 0 \pmod 2$, and hence if and only if q is of the form $8n\pm 1$. But q is

of the form $8n \pm 1$ if and only if $(-1)^{(q^2-1)/8} = 1$, and this with the above result gives

$$(2 \mid q) = (-1)^{(q^2-1)/8},$$

the second supplement to the reciprocity law.

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A SYMBOLIC FORM OF THE CLASSICAL COMPLEX INVERSION FORMULA FOR A LAPLACE TRANSFORM

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In this journal [1] the author showed that an inversion formula, due to E. Phragmén, for the Laplace transform

$$F(s) = \int_0^\infty e^{-st}\Phi(t)dt \tag{1}$$

is symbolically equivalent to $\{1/\Gamma(D)\}\ F(e^{-z})$, where D stands for differentiation with respect to x and $\Gamma(s)$ is the Gamma-function. In an earlier note [2] the same result was established for the Post-Widder inversion of (1). However, it seems not to have been pointed out that this same symbolic operator $1/\Gamma(D)$ also yields the classical [3] complex inversion

$$\Phi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds, \qquad 0 < t < \infty. \tag{2}$$

Here the path of integration is the vertical line $\sigma = c$ in the plane of the complex variable $s = \sigma + i\tau$. The constant c must be greater than σ_c, the abscissa of convergence for the integral (1); and the Cauchy principal value for the integral (2) may be needed.

For completeness let us sketch heuristically how one may arrive at the above operator. We remind the reader that $e^{aD}f(x)$ is interpreted as $f(x+a)$ in view of the MacLaurin expansion

$$e^{aD} = \sum_{n=0}^{\infty} \frac{a^n D^n}{n!},$$

$$e^{aD}f(x) = \sum_{n=0}^{\infty} \frac{a^n}{n!} f^{(n)}(x) = f(x+a). \tag{3}$$

Now if we make an exponential change of variable, equation (1) becomes

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\phi(t)dt = \int_{-\infty}^{\infty} G(t)\phi(x-t)dt, \tag{4}$$

where

$$G(x) = e^{-e^{-x}}, \qquad f(x) = F(e^{-x}), \qquad \phi(x) = \Phi(e^x)e^x.$$

From the definition of the Gamma-function we have

$$\Gamma(D) = \int_0^{\infty} t^{D-1}e^{-t}dt = \int_{-\infty}^{\infty} e^{-tD}G(t)dt.$$

By use of (3) and (4) we consequently obtain

$$\Gamma(D)\phi(x) = \int_{-\infty}^{\infty} \phi(x-t)G(t)dt = f(x),$$

and a formal solution of this equation yields the desired symbolic inversion formula

$$\phi(x) = \frac{1}{\Gamma(D)} f(x). \tag{5}$$

The appropriate representation of $1/\Gamma(s)$ for the present purposes is one due to Laplace [4],

$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^z z^{-s}dz, \qquad c > 0, \sigma > 0.$$

Replacing s by D and again using (3) we obtain

$$\frac{1}{\Gamma(D)} f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^z f(x - \log z)dz.$$

In terms of the original functions equation (5) becomes

$$\Phi(e^x)e^x = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^z F(ze^{-x})dz,$$

or, if $e^x = t$,

$$\Phi(t)t = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^z F\left(\frac{z}{t}\right)dz \qquad \frac{c}{t} > \sigma_c.$$

This equation is equivalent to (2) if $z = ts$. The path of integration becomes $\sigma = tc$, but by Cauchy's theorem a single vertical line may be used for all t.

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A SIMPLE EVALUATION OF AN IMPROPER INTEGRAL

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The purpose of this note is to give a simple proof of the relation

$$\int_0^{+\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}. \tag{1}$$

Our proof of (1) is based on the following theorem due to Riemann: *If $f(x)$ is Riemann integrable in the interval $a \leqq x \leqq b$, then*

$$\lim_{k\to\infty} \int_a^b f(x) \sin kx\,dx = 0. \tag{2}$$

To proceed with our proof, we may notice that

$$\int_0^{\pi} \frac{\sin (n+\frac{1}{2})x}{2 \sin \frac{x}{2}}\,dx = \frac{\pi}{2}, \qquad (n = 0, 1, 2, \cdots). \tag{3}$$

The relation (3) is obtained by integrating both sides of the identity

$$\tfrac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx = \frac{\sin (n+\frac{1}{2})x}{2 \sin \frac{x}{2}}$$

from 0 to π.

Let

$$\phi(x) = \frac{1}{x} - \frac{1}{2 \sin \frac{x}{2}} = \frac{2 \sin \frac{x}{2} - x}{2x \sin \frac{x}{2}}, \qquad 0 < x \leqq \pi. \tag{4}$$

The function $\phi(x)$ is continuous for $0 < x \leqq \pi$. By applying twice L'Hospital's rule to (4), we see that

$$\lim_{x\to 0} \phi(x) = 0.$$

Consequently, if we put $\phi(0) = 0$, $\phi(x)$ will be continuous for $0 \leqq x \leqq \pi$. Therefore $\phi(x)$ certainly satisfies the condition of the Riemann theorem; then the rela-

tion (2), with $k=n+\frac{1}{2}$, gives

$$\lim_{n\to\infty}\int_0^{\pi}\left[\frac{1}{x}-\frac{1}{2\sin\frac{x}{2}}\right]\sin(n+\tfrac{1}{2})x\,dx=0.$$

Taking into account (3), we see that

$$\lim_{n\to\infty}\int_0^{\pi}\frac{\sin(n+\frac{1}{2})x}{x}\,dx=\frac{\pi}{2},$$

or, making the substitution $u=(n+\frac{1}{2})x$,

$$\lim_{n\to\infty}\int_0^{(n+1/2)\pi}\frac{\sin u}{u}\,du=\frac{\pi}{2}. \tag{5}$$

Since

$$\int_0^{+\infty}\frac{\sin u}{u}\,du$$

is convergent, (5) yields the relation (1).

CLASSROOM NOTES

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FUNDAMENTALS IN THE TEACHING OF UNDERGRADUATE MATHEMATICS*

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1. Introduction. What I have to say today is neither new nor startling. It has all been said before many times both by others and by myself.** I undertook to say it again because, judging by my own behavior in the classroom, frequent repetitious prodding is needed to keep one on the best path whenever the best path does not coincide with the easiest. Teaching mathematics well at the undergraduate level is not an easy task. I hasten to add that what I mean by teaching "well" may not be what others mean by it; and I do not insist that those who disagree with me are wrong by definition. In fact, I believe that diversity in points of view is healthy and should be encouraged by allowing individual instructors maximum freedom rather than impose a stifling uniformity.

* Presented before Section 7 of the International Congress of Mathematicians at Cambridge, Mass., September 1, 1950.

** Cf. bibliography at the end of the paper.

In passing, let me mention one troublesome feature of undergraduate teaching which can be largely overcome, namely the ever present necessity for compromise due to the fact that it is frequently impossible, at that level, to tell the whole truth. A good cure for this is to try to tell nothing but the truth, even at the cost of restricting generality rather than to use faulty arguments to bolster general truths. If a theorem has to be used which is too hard to prove, then one should assume it or give an informal heuristic discussion rather than pass off an incorrect proof as correct.

In general, *I deplore the tendency to claim that mathematics stimulates clear thinking and then to teach it as meaningless memorized routine techniques having little to do with thought.* This tendency is unfortunately aided and abetted by many textbooks. It is admittedly difficult to compel oneself to think carefully in an elementary class which one teaches perennially, or when writing an elementary textbook. For example, how many textbooks on analytic geometry even mention the fact that a converse proposition is necessary in establishing an equation of a locus such as an ellipse† or hyperbola? Converses are traditionally stressed in secondary school classes in plane geometry and frequently forgotten thereafter. Or, how many calculus texts refrain from stating without qualification the half-truth that

$$\int \frac{dx}{x} = \ln x + C? \tag{1}$$

This is not, to my mind, mere pedantry. Evil consequences may attend any lie, no matter how small. For example, (1) causes the loss of a whole family of curves in one of the simplest examples in a first course in differential equations. From the family of curves $xy=c(c\neq 0)$, we get the differential equation $dx/x+dy/y=0$. Integrating, (1) yields $\ln x+\ln y=\ln k(k>0)$, or $xy=k(k>0)$. Where have the curves $xy=k(k<0)$ gone? The correct formula‡

$$\int \frac{dx}{x} = \ln |x| + C \tag{2}$$

yields, of course, all the curves $|xy|=k$ or $xy=\pm k(k>0)$. Few students, in my experience, can locate the lost curves because the incomplete formula (1) is so strongly entrenched in our textbooks.

Time does not permit further examples. The point I wish to make is that *many ills can be traced to excessive formalism in instruction.*

2. Proper emphasis in the teaching of undergraduate mathematics. Every course in mathematics touches to some extent each of the following three

† For a correct treatment, see, for example, Osgood and Graustein, Plane and Solid Analytic Geometry, Macmillan 1930.

‡ Cf. R. Courant, Differential and Integral Calculus, vol. 1 (Nordemann 1937), translated by E. J. McShane, for the derivation.

aspects of the subject:

T. Techniques;
A. Applied problems;
F. Fundamentals.

Under techniques I include the formal algorithmic aspects of mathematics. Under fundamentals I include fundamental concepts, careful reasoning including reasonable justification of the techniques, logical structure, historical evolution of ideas, the relevance of mathematics to other subjects and its cultural significance including justification of its position at the base of the Tree of Knowledge.*

In the practice of teaching, T is too often overstressed to the detriment of A, which runs a poor second, and the virtual exclusion of F which receives little but lip-service. In my opinion, these three aspects should be equally stressed for science or mathematics majors, while for majors in other divisions the order should be exactly reversed.

In recent years, progress has been made in creating terminal courses for non-science and non-mathematics majors which stress F and A while limiting T to bare essentials. These relatively new courses differ in details but share, I think, the principal general objectives.

However, today I wish to concern myself with science and mathematics majors, including prospective teachers of mathematics at secondary school and junior college levels. For this class of students, the problem of successfully teaching F without diminishing the desired attention to T and A remains a difficult but important one. Too many in this class emerge from undergraduate instruction equipped with a collection of heterogeneous techniques, a smattering of applications, and little or no grasp of fundamentals. One consequence of this lack is the student's inability to teach or defend his subject intelligently. Truly, mathematics is a study often pursued but seldom overtaken. I am convinced that many intelligent colleagues in other departments who wish to remove mathematics from the list of prescribed courses have misunderstood the value of the subject because it is so often presented as senseless memorized drill rather than with emphasis on its reasonableness and relevance to the real world. They have themselves been exposed to coaching courses for stereotyped examinations rather than to genuine courses in mathematical thinking. Needless to say, unrationalized teaching of formal techniques is also a source of many technical errors, such as the "Universal Cancellation Law" and the "Universal Distributive Law." The first of these says that a symbol may be crossed out in both places if it appears twice in the same expression, as in, $\sin x/\sin y = x/y$, $\sin x = \sin y$ implies $x = y$, etc. Familiar examples of the latter are $\log (x+y) = \log x + \log y$, $\sin (x+y) = \sin x + \sin y$, $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$, $D(f \cdot g) = Df \cdot Dg$, etc.

* This justification is intended here not merely in the sense of mathematics as the science of space and quantity but also in the sense of mathematics as the totality of hypothetico-deductive sciences and their concrete interpretations.

3. Reasons why fundamentals are understressed. The lack of stress on fundamentals is to my mind the outstanding weakness of much contemporary teaching of undergraduates both in mathematics and in natural science. The liberating effect of scientific rationalism which was felt so strongly in preceding centuries seems now to be lost in the authoritarian dicta of the instructor and the undigested, regurgitated rote responses of the student. To what causes can we attribute this sorry state of affairs?

(a) A few teachers seem to feel that fundamentals should not be taught to most undergraduates. They act as if they believed that it is some how improper for young students to think but entirely proper for them to learn tricks of calculation blindly. Those who hold this extreme view are possibly authoritarians by nature and their number is, fortunately, not great.

(b) Some teachers believe that fundamentals should be taught to undergraduates but that it cannot be done successfully. These are inclined to render superficial lip-service to fundamentals and hurry on to formal techniques with which the student is apt to have easier success.

(c) It is admittedly more difficult to teach F than either T or A. F demands of the teacher resources of patience, eloquence, and energy far greater than those needed for T or A. Being a drill-master and stressing memorized mastery of routine algorithms makes the teacher's life easier but harder to justify.

(d) It is difficult to test and, especially, to grade results of time spent on F.

(e) T and A leave little time free for F. In fact, neglect of F is often due to the teacher's anxiety to "cover" ground in T and A. However, ground can be "covered" as with an opaque blanket as a result of this anxiety. I am convinced that even a routine technique will be remembered longer and applied better if it is understood than if it is merely memorized.

4. Suggested remedies. I have no panacea for these conditions. However, the following suggestions are offered as partial remedies.

(a) Adequate time should be allowed in every course for discussion of fundamentals. The student will learn best whatever the teacher stresses. If the teacher indicates by his attitude that he doesn't expect the student to learn something or that he doesn't consider it important, the average student can scarcely be expected to grasp it.

(b) Effort should be made to include some questions on fundamentals on examinations. It is hardly necessary to labor this obvious point.

(c) Supplementary reading is a device which is not sufficiently used in mathematics courses although it is traditional in other departments. I have personally known able students who never looked at a book other than the textbook for the course they were taking. On the other hand, I have received gratifying expressions of thanks from others for guiding them to some of the good literature of mathematics of whose existence they had been unaware. In fact, several mathematics majors have told me that they derived as seniors considerable enjoyment and enlightenment from some of the books written for

non-science freshmen. This phenomenon is in itself evidence supporting my contention that fundamentals are understressed.

(d) Special courses in fundamentals at an elementary level may be of value especially for prospective teachers of secondary and junior college mathematics who are not ordinarily exposed to extensive graduate study in mathematics. I would consider such a course of greater value to this class of students than one more technical course in advanced mathematics.

5. Conclusion. *The main need, as I see it, is for teachers to be so convinced of the value of fundamentals that they will push themselves along that path even though it is not the path of least effort.*

Mathematics is more than a collection of tricks for calculating and solving problems. Its ideas constitute a main stream in the development of human thought. It is characterized not by sly devices which mysteriously seem to work but rather by persistent and complete honesty in its explicit recognition of underlying assumptions and in its adherence to logical proof of its theorems and precision in its use of terms. How few students come away from their undergraduate studies with a firm grasp of these ideals! Only the more gifted students will develop really independent insight and creative imagination; but the average student *can* understand in large measure the standards of critical thinking. The realization that theories depend on their underlying hypotheses and that various assumptions may be equally tenable is in itself a long step toward maturity and tolerance. I would measure the real success of undergraduate teaching of mathematics at least as much by the extent to which these fundamental ideas are instilled in the student as by the efficiency with which he can perform routine techniques such as Horner's method, complicated formal integrations and the like.

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REMARKS ON NATURAL NUMBERS

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The authors' somewhat unorthodox yet rational approach to the system of natural numbers (see, *On the System of Natural Numbers*, this MONTHLY, vol. 57, 1950, pp. 104–112) has aroused certain criticisms (see, Rosser, *Problems on*

Natural Numbers, this MONTHLY, vol. 57, 1950, pp. 333–334). We wish to make a few general remarks for the purpose of clarifying the point of view intended in our essay and to correct some minor errors and omissions which have been noted in it. All references are to the authors' article cited above unless otherwise stated.

Since our essay is directed to the elementary student, we let constructive intuitition play an important role at the beginning. The postulates are to be considered as significative, mathematically relevant assumptions about the intuitive notions and are not to be regarded (as logicians usually do) as indirect definitions of the undefined terms, such terms having no meaning other than that implied by the postulates. The method of definitions by postulates requires the consistency proof and this has never been satisfactorily demonstrated for the fundamentals of arithmetic and perhaps cannot ever be attained. In our genetic type of approach the *existence* of things or relations described by the primary concepts (unity, set, belonging to a set, etc.) is not postulated but regarded as a matter of fact, as a starting point, as necessary knowledge prior to our development.

Postulates 1 and 2, on the other hand, introduce in a simple way the idea of infinity. They are intended to apply to proper sets as stated in Section 2. There is need, however, to allow one exception in Postulate 2, which may be specified by splitting the postulate as follows:

POSTULATE 2. (a) *Given a set X, it is possible to form another set by adjoining another element to those of X;*

(b) *Given a set X, not a unit set, it is possible to form another set by removing an element from X;*

(c) *Given a unit set X_u, it is possible to remove its single element and thus obtain the null set.*

Postulate 3 introduces in an elementary way the idea of free choice or selection. We accepted the point of view expressed by Hobson (*The Theory of Functions of a Real Variable*, 2nd edition, Vol. 1, pp. 248–252) in which he points out that it may be necessary in certain situations to invoke the axiom of Zermelo in the case of only a finite number of sets from which the selection is to be made. At this primitive stage of the development we regarded Postulate 3 as establishing the possibility of selecting a single element from each of two sets, without necessary reference to the logical possibility of an effective determination of such elements. In this sense, Postulate 3 is a *case* of the Zermelo Axiom, that is, applied to two sets.

Nevertheless, we grant that our offhand reference to Postulate 3 as *the* axiom of Zermelo is not literally true, and it should have been made sufficiently clear that we meant it as the axiom of choice for the case in hand. However, it is likewise not literally true that this notorious axiom refers only to an infinite number of sets from which the selection is to be made.

Postulate 4 does characterize linear order *in the class of ordered sets* as introduced in the last paragraph of Section 2, that is, those sets for which a rule is

given to decide for any two distinct elements of the set which is the preceding. However, it is probably not sufficiently clear from our statement of Postulate 4 that it is to apply only to ordered sets as previously discussed and it would be better to have the fourth property of linear order explicitly stated in it.

Definition 5 should read: "Elements A and B, A preceding B, of an ordered set are said . . . ."

In the first sentence of Section 11 the product ab is defined with the stipulation $b>1$. Thus $a\cdot 0$ and $a\cdot 1$ require further definitions. We considered the sum of sets as a set operation or function on two or more arguments. This should have been pointed out in Section 10. It is true that multiplication of natural numbers can be defined in terms of sets independently of addition in such a way as to cover all cases if the use of the null set is allowed. In this case it is not necessary to decide whether a sum can have fewer than two summands. However, we decided against this procedure in our elementary treatment.

In the proof of the principle of complete induction on pages 107 and 108, the element K has an immediate predecessor J since by hypothesis K cannot be the first element of the set. This is clearly implied by the phraseology used in the proof.

The theorem on top of page 108 follows from the induction on the elements of a set Y whose elements are the subsets $X', \cdots, X'', X''', \cdots, X$ of the original set (for examples of similar proofs, see, Hobson, Loc. cit., pp. 4–8). Since this proof involves a particularly defined notation, perhaps the following indirect proof is more cogent. Let X be a given finite set, not a unit set. Let x_1 denote its first element (Definition 6) and remove it. If the *residual* set X_1, which is finite, is not a unit set, let x_2 denote its first element. By removing the element x_2 from X_1, we obtain a finite residual set X_2. Proceeding in this manner, if we do not reach a unit set X_u, we will have found a subset $(x_1, x_2, \cdots)$ of X which has no last term, which contradicts the hypothesis that X is finite. Since a unit set can be made empty by Postulate 2 above, the theorem is proved.

A SIMPLE ENDPOINT MINIMUM

FRANK HAWTHORNE, Hofstra College

C. O. Oakley (this MONTHLY, vol. 54, p. 407) has discussed endpoint maxima and minima with examples. A particularly simple example which may be used to illustrate this point is:

Find that point on the circle $x^2+y^2=1$ which is nearest to (2, 0).

The unsuspecting student sets up this problem using x as independent variable and obtains, for the square of the distance from (x, y) to (2, 0),

$$L^2 = (2 - x)^2 + 1 - x^2 = 5 - 4x.$$

Differentiating and equating the result to zero he obtains the disconcerting expression $-4=0$. This certainly does not lead to the obvious solution (1, 0).

If, however, the student uses the central angle as independent variable this difficulty is avoided.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTIONS

E 956. *Proposed by H. F. Sandham, Trinity College, Ireland*

Let a_0 and b_0 be positive numbers and define a_{n+1} and b_{n+1} as the arithmetic and harmonic means, respectively, of a_n and b_n. Show that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = (a_0 b_0)^{1/2}.$$

E 957. *Proposed by C. O. Oakley, Haverford College*

A *palindrome* is an integer that reads the same forwards as backwards as, for example, 2332. By definition the ten digits 0, 1, 2, $\cdots$, 9 will be considered palindromes. How many palindromes are there of at most n digits?

E 958. *Proposed by Herbert Scarf, Temple University*

If $a_n = 1/n!$ show that

$$\begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_n & \cdot & \cdot & \cdot & \cdots & a_1 \end{vmatrix} = a_n.$$

E 959. *Proposed by C. S. Ogilvy, Columbia University*

Let C_0 be a square, C_{n+1} the convex polygon obtained from C_n by measuring off one fourth of the length of every side of C_n and cutting off the corners, and $C = \lim_{n\to\infty} C_n$. Prove that C consists of four parabolic arcs.

E 960. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let AB be a diameter of a circle (O) and let C and D be the midpoints of the radii OA, OB. Let the circles $A(O)$, $B(O)$ cut (O) in A_1, B_1 on the same side of AB. On the other side of AB draw the semicircles $C(O)$ and $D(O)$. Consider the sequence of circles (O_1), (O_2), (O_3), $\cdots$, (O_n), $\cdots$ tangent to the arcs OA_1, OB_1 of $A(O)$ and $B(O)$ and, in turn, to (O), (O_1), (O_2), $\cdots$, (O_{n-1}), $\cdots$. Also consider the sequence of circles (O_1'), (O_2'), (O_3'), $\cdots$, (O_n'), $\cdots$ tangent to the semicircles $C(O)$ and $D(O)$ and, in turn, to (O), (O_1'), (O_2'), $\cdots$, (O_{n-1}'), $\cdots$. Show that the sum of the lengths of the circumferences of all the circles (O_n) and (O_n') is equal to the length of the circumference of (O).

SOLUTIONS

Possible Number of Edges for a Polyhedron

E 923 [1950, 416]. *Proposed by R. C. Buck, University of Wisconsin*

Show that no polyhedron in three-space can have exactly seven edges, while any other integer greater than five is admissible.

Solution by E. P. Starke, Rutgers University. Let $n>3$. A simple polyhedron of $2n$ edges is a pyramid with an n-gon for its base. If the n-gon is folded along a diagonal so that it lies in two planes before its vertices are joined to a point outside the two planes, the resulting polyhedron has $2n+1$ edges.

Each vertex of a polyhedron is also the vertex of a polyhedral angle with at least three edges, and each edge of the polyhedron serves as an edge of two polyhedral angles. A polyhedron with four vertices is a tetrahedron and has six edges. Every other polyhedron has five or more vertices, hence at least (5) (3) edges to its polyhedral angles, and hence at least (5)(3)/2 edges. Thus seven edges is impossible.

Also solved by Norman Anning, F. Bagemihl, Alan Berndt, Ray Jurgensen, H. C. Kranzer, Roger Lessard, Norman Miller, L. A. Ringenberg, Harold Shniad, C. W. Trigg, and the proposer.

Most of the above solutions assumed the polyhedron to be simply-connected, and hence subject to the Euler-Descartes formula: $v+f=e+2$. Bagemihl pointed out that this case was solved by Euler in *Elementa doctrinae solidorum*, Novi comm. acad. sc. imp. Petropolitanae 4 (1752), pp. 109–140.

Anning located the problem as No. 7 of Challenge Exercise 63 in H. G. Forder's *School Geometry*, p. 285. On p. 260 of his *Higher Course Geometry*, Forder gives a solution to the problem.

The proposer noted that for the corresponding problem in n-space no value of $e<n(n+2)/2$ is admissible except $n(n+1)/2$.

An Interesting Limit

E 924 [1950, 416]. *Proposed by D. J. Newman, New York University*

Find $\lim_{n\to\infty} n \sin (2\pi en!)$.

I. *Solution by F. Bagemihl and W. Seidel, University of Rochester.* Using the familiar series for e and $\sin x$ and the periodicity of the latter, we have

$$\begin{aligned} n \sin (2\pi en!) &= n \sin \left[2\pi n! \sum_{k=0}^{\infty} 1/k! \right] \\ &= n \sin \left[2\pi \sum_{k=1}^{\infty} 1/(n+1)(n+2)\cdots(n+k) \right] \\ &= 2\pi n/(n+1) + O(1/n). \end{aligned}$$

Hence

$$\lim_{n\to\infty} n \sin (2\pi en!) = 2\pi.$$

II. *Solution by C. H. Murphy, Chevy Chase, Md., and Eugene Usdin, Purdue University.* As above

$$n \sin (2\pi en!) = n \sin (2\pi R_n),$$

where

$$R_n = \sum_{k=1}^{\infty} 1/(n+1)(n+2)\cdots(n+k).$$

Now

$$1/(n+1) < R_n < \sum_{k=1}^{\infty} 1/(n+1)^k = 1/n.$$

Therefore

$$\lim_{n\to\infty} R_n = 0 \quad \text{and} \quad \lim_{n\to\infty} nR_n = 1,$$

whence

$$\begin{aligned}\lim_{n\to\infty} n \sin (2\pi en!) &= \lim_{n\to\infty} n \sin (2\pi R_n) \\ &= \lim_{n\to\infty} 2\pi nR_n[\sin (2\pi R_n)/2\pi R_n] \\ &= 2\pi.\end{aligned}$$

Also solved by P. M. Anselone, S. D. Berkowitz, N. J. Fine, P. G. Kirmser, H. C. Kranzer, Norman Miller, L. A. Ringenberg, J. W. Ross, Harold Shniad, O. E. Stanaitis, R. W. Wild, A. W. Wortham, and the proposer.

M. Golomb noted that this limit also proves that e is irrational. For if e was rational then $n!e$ would be integral for all sufficiently large n, and consequently we would have

$$\lim_{n\to\infty} n \sin (2\pi en!) = 0.$$

Weighted Dice

E 925 [1950, 416]. *Proposed by J. B. Kelly, University of Wisconsin*

Is it possible so to weight a pair of dice that the probability of occurrence of every sum from 2 to 12 shall be the same?

I. *Solution by Leo Moser and J. H. Wahab, University of North Carolina.* Let p_i and q_i denote the probabilities of an i appearing on the first and second die, respectively. Let s_j denote the probability of the occurrence of the sum j. Now assume $s_2 = s_{12}$. Then

$$p_1q_1 = s_2 = s_{12} = p_6q_6,$$

whence

$$(p_1 - p_6)(q_1 - q_6) \leqq 0,$$

so that

$$s_2 + s_{12} = p_1q_1 + p_6q_6 \leqq p_1q_6 + p_6q_1 \leqq s_7.$$

Thus no loading of the dice can yield equiprobable sums.

II. *Solution by J. V. Finch and P. R. Halmos, University of Chicago.* Define p_i and q_i as above. If $P(x) = \sum_{i=0}^{5} p_{i+1}x^i$ and $Q(x) = \sum_{i=0}^{5} q_{i+1}x^i$, then the requirement that all sums (between 2 and 12) are equally probable is equivalent to the identity $P(x)Q(x) = (1/11)\sum_{i=0}^{10} x^i$. The zeros of the polynomial on the right are all complex (they are, in fact, the complex 11th roots of unity) whereas either of the factors on the left (having real coefficients and odd degree) has at least one real root. It follows that no such factoring is possible and therefore that no loading of the dice can yield equiprobable sums.

Also solved by G. M. Dillon, N. J. Fine, Daniel Finkel, Ray Jurgensen, H. C. Kranzer, Roger Lessard, Bart Park, Harold Shniad, P. B. Wood, A. W. Wortham, and the proposer.

Four Terms in Arithmetic Progression

E 926 [1950, 483]. *Proposed by H. L. Lee, University of Tennessee*

Show that there exist arithmetic progressions with integral terms such that the sum of the squares of four consecutive terms is a 2^kth power of an integer, k a non-negative integer.

I. *Solution by S. T. Thompson, Tacoma, Washington.* We establish a slightly more general result. Let $a-d$, a, $a+d$, $a+2d$ be four consecutive terms of an arithmetic progression. If we take

$$a = 6^{n-1}, \qquad d = 2a, \qquad n > 0,$$

we find that

$$(a - d)^2 + a^2 + (a + d)^2 + (a + 2d)^2 = 36a^2 = 36^n.$$

The given problem is settled, for $k>0$, by taking $n=2^{k-1}$; it is trivial for $k=0$.

II. *Solution by K. Subbarao, M. R. College, Vizianagram, India.* Let $a-3d$ be the first of the four consecutive terms of the arithmetic progression with common difference $2d$. Then

$$(a - 3d)^2 + (a - d)^2 + (a + d)^2 + (a + 3d)^2 = (2a)^2 + 5(2d)^2 = x^2 + 5y^2,$$

where x and y are even. Setting

$$x = p^2 - 5q^2, \qquad y = 2pq, \qquad p, q \text{ even},$$

we have

$$x^2 + 5y^2 = (p^2 + 5q^2)^2.$$

Again, setting

$$p = m^2 - 5n^2, \qquad q = 2mn, \qquad m, n \text{ even},$$

we have

$$x^2 + 5y^2 = (p^2 + 5q^2)^4.$$

Repetition of this process leads to the desired result.

Following are some theorems related to the given problem.

1. The Diophantine equation

$$x_1^2 + x_2^2 + \cdots + x_{2p+1}^2 = (2p + 1)z^n,$$

where $x_1, x_2, \cdots, x_{2p+1}$ are in arithmetic progression and $3p(p+1)$ is non-square, has infinitely many solutions.

2. The Diophantine equation

$$x_1^2 + x_2^2 + \cdots + x_{2p}^2 = 2pz^n,$$

where $x_1, x_2, \cdots, x_{2p}$ are in arithmetic progression and $3(4p^2-1)$ is non-square, has infinitely many solutions.

3. The Diophantine equation

$$x_1^2 + x_2^2 + \cdots + x_m^2 = z^{2n+1},$$

where $x_1, x_2, \cdots, x_m$ are in arithmetic progression, has infinitely many solutions.

4. The Diophantine equation

$$x_1^3 + x_2^3 + \cdots + x_m^3 = z^n,$$

where $x_1, x_2, \cdots, x_m$ are in arithmetic progression, has infinitely many solutions for n of the form $3s+1$ or $3s+2$.

Also solved by D. H. Browne, C. A. Johnson and S. J. Pagano (jointly), M. S. Klamkin, Roger Lessard, W. V. Parker, R. E. Wild, and the proposer.

An Unsolved Problem

E 534 [1942, 475, 1943, 261, 1950, 557]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Show that 4, 5, 7 are the only values of n for which $n!+1$ is a perfect square.

Comment by H. D. Grossman, New York, N. Y. The solution that we offered for this problem [1950, 557] contains an invalidating oversight. We cannot conclude, as stated in the third paragraph, that 6, 10, 14, $\cdots$ divide s. Be it resolved that E 534 is still "An Unsolved Problem."

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscript should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4428. *Proposed by C. D. Olds, San Jose State College, California*

If p is any prime number >3, prove that

$$kC_{p-1} \equiv jC_{p-1} \pmod{p^3},$$

where j, k are any integers and ${}_nC_r$ is the number of combinations of n different things taken r together.

4429. *Proposed by Ky Fan, University of Notre Dame*

Let H be a non-negative (positive semi-definite) Hermitian transformation in the n-dimensional unitary space. If the eigenvalues $\lambda_i (1 \leqq i \leqq n)$ of H are arranged in ascending order: $0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$, show that, for any positive integer $k \leqq n$, the product $\lambda_1 \lambda_2 \cdots \lambda_k$ of the first k eigenvalues is equal to the minimum of the expression

$$\prod_{i=1}^{k} (Hx_i, x_i),$$

when k orthonormal vectors $x_1, x_2, \cdots, x_k$ vary in space. Here (Hx_i, x_i) denotes the inner product of the vector Hx_i with x_i. (This result is related to the Proposer's paper, *On a theorem of Weyl concerning eigenvalues of linear transformations*, I, Proceedings of the National Academy of Sciences, U. S. A., v. 35 (1949), pp. 652–655.)

4430. *Proposed by Ky Fan, University of Notre Dame.*

Let H', H'' be two n-rowed non-negative Hermitian matrices, and let $H = \alpha H' + \beta H''$, where α, β are two non-negative numbers with $\alpha + \beta = 1$. Let the eigenvalues of H', H'' and H be denoted by λ_i', λ_i'' and $\lambda_i (1 \leqq i \leqq n)$ respectively and so arranged that

$$\lambda_i' \leqq \lambda_{i+1}', \qquad \lambda_i'' \leqq \lambda_{i+1}'', \qquad \lambda_i \leqq \lambda_{i+1}.$$

Prove that for any positive integer $k \leqq n$:

$$\lambda_1 \lambda_2 \cdots \lambda_k \geqq (\lambda_1' \lambda_2' \cdots \lambda_k')^{\alpha} (\lambda_1'' \lambda_2'' \cdots \lambda_k'')^{\beta}, \tag{1}$$

in particular:

(2) $$\text{Det } H \geqq (\text{Det } H')^{\alpha}(\text{Det } H'')^{\beta}.$$

4431. *Proposed by M. S. Klamkin, Brooklyn Polytechnic Institute, New York*

If $S_n = \sum_1^n 1/r$, prove:

$$\sum_{n=1}^{\infty} \frac{S_n}{n^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^3} = 2\sum_{n=1}^{\infty} \frac{S_n}{(n+1)^2}.$$

4432. *Proposed by Victor Thébault, Tennie, Sarthe, France*

With the midpoints of the sides of a triangle as centers circles are described passing through the feet of the corresponding altitudes. Show that the circle orthogonal to these three circles is tangent to the nine-point circle of the given triangle at a point such that its distance from the foot of one altitude is equal to the sum of the distances from the feet of the other two altitudes.

SOLUTIONS

Professor Banach's Matches

4348 [1949, 343]. *Proposed by D. A. Darling, Rutgers University*

This problem was brought from Poland by Professor H. Steinhaus. It appears that Professor Banach was accustomed to carrying a box of matches in each of two coat pockets. To light his pipe, he would take a match from either box at random. The boxes contained originally n matches each. Banach's question is: when first a box is opened and found empty, what is the expected number of matches left in the other box?

Solution by W. D. Berg, Kenyon College, Gambier, Ohio. The probability of drawing a particular box exactly n times in $n+x$ ($x=0, 1, \cdots, n$) trials and drawing it again on the $(n+x+1)$th trial is $\binom{n+x}{n}(\frac{1}{2})^{n+x+1}$. Therefore when first a box is opened and found empty the probability that $n-x$ matches remain in the other box is

$$2\binom{n+x}{n}(\tfrac{1}{2})^{n+x+1} = \binom{n+x}{n}(\tfrac{1}{2})^{n+x}.$$

Since this situation must obtain for one and only one value of x, we note that

(1) $$\sum_{x=0}^{n}\binom{n+x}{n}(\tfrac{1}{2})^{n+x} = 1.$$

The expected number E of matches left in the other box is therefore

$$E = \sum_{x=0}^{n}(n-x)\binom{n+x}{n}(\tfrac{1}{2})^{n+x},$$

which, with the use of (1) may be simplified as follows.

$$E = (2n+1)\sum_{x=0}^{n}\binom{n+x}{n}(\tfrac{1}{2})^{n+x} - \sum_{x=0}^{n}(n+1+x)\binom{n+x}{n}(\tfrac{1}{2})^{n+x}$$

$$= (2n+1) - 2(n+1)\sum_{x=0}^{n}\binom{n+1+x}{n+1}(\tfrac{1}{2})^{n+1+x}$$

$$= (2n+1) - 2(n+1)\left\{1 - \binom{2n+2}{n+1}(\tfrac{1}{2})^{2n+2}\right\}$$

$$= \frac{(2n+1)!}{n!n!}(\tfrac{1}{2})^{2n} - 1.$$

Also solved by Colin Blyth, Herbert Carus, J. A. Clarkson, E. H. Cutler, Jacques Dutka, Solomon Leader, Frederick Mosteller, Bart Park, C. F. Pinzka, Lois B. Whitman, and the proposer. Bernard Friedman, R. E. Greenwood, Fritz Herzog, J. G. Millar, Leo Moser, and Frederick Mosteller found the answer $\binom{2n}{n}n/2^{2n-1}$ for the allied problem: What is the expected number of matches remaining when one of the boxes is first emptied.

Editorial Note. Several solvers point out that E is asymptotically equivalent to $2\sqrt{n/\pi}$. Millar observes that E is one less than the coefficient of x^n in the expansion of $(1-x)^{3/2}$. Five readers sent in replies based on the unwarranted assumption that all applicable permutations of $n+1$ A's and $n+1$ B's (where A, B represent a drawing from one box or the other) are equally likely.

Dutka notes that the problem is quoted and solved in Feller, *An Introduction to Probability Theory and Its Applications*, New York, 1950, pp. 108–109, 176–177. Mosteller points out its similarity to that of average sample size in truncated single sampling in industrial sampling inspection. See, for example, Statistical Research Group of Columbia University, *Sampling Inspection*, McGraw-Hill, 1948. In industrial inspection plans the n's would ordinarily be unequal.

An Infinite Continued Ratio

4349 [1949, 344]. *Proposed by H. F. Sandham, Trinity College, Dublin, Ireland*

Prove that

$$\frac{2}{1}\Big/\frac{5}{4}\Big/\frac{8}{7}\Big/\frac{11}{10}\cdots = \sqrt{3}.$$

I. *Solution by J. B. Kelly, University of Wisconsin.* From the well known identity

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

one obtains

$$\frac{z}{w}\prod_{n=1}^{\infty}\frac{(n-z)(n+z)}{(n-w)(n+w)}=\frac{\sin \pi z}{\sin \pi w}.$$

In particular, the substitution $2z=a$, $2w=1-a$ gives a form which reduces to

$$\frac{a}{1-a}\Big/\frac{1+a}{2-a}\Big/\frac{2+a}{3-a}\Big/\frac{3+a}{4-a}\cdots=\tan\frac{\pi a}{2}.$$

The proposed formula is the special case where $a=2/3$.

II. *Solution by S. T. Parker, Kansas State College.* The problem may be generalized to provide similar developments for other radicals. The fraction

$$F(a)=\frac{2\cdot 4\cdot 6\cdots 2a}{1\cdot 3\cdot 5\cdots(2a-1)}=\frac{2^{2a}a!a!}{(2a)!}$$

diverges as the integer a increases without limit, but the quotient

$$\frac{F(na)}{F(a)}=\frac{2^{2a(n-1)}(na)!(na)!(2a)!}{a!a!(2na)!} \tag{1}$$

converges to the limit $\sqrt{n}$, as is seen at once from Stirling's formula. For n odd, if each factor in $F(a)$ is multiplied by n, (1) is equivalent to $F(na)$ with all multiples of n removed. It can then be easily expanded as a continued ratio $r_0/r_1/r_2\cdots$, where

$$r_k=\frac{(kn+2)(kn+4)\cdots(kn+n-1)}{(kn+1)(kn+3)\cdots(kn+n-2)}.$$

For example, the present problem and

$$\sqrt{5}=\frac{2\cdot 4}{1\cdot 3}\Big/\frac{7\cdot 9}{6\cdot 8}\Big/\frac{12\cdot 14}{11\cdot 13}\Big/\frac{17\cdot 19}{16\cdot 18}\cdots,$$

$$\sqrt{7}=\frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5}\Big/\frac{9\cdot 11\cdot 13}{8\cdot 10\cdot 12}\Big/\frac{16\cdot 18\cdot 20}{15\cdot 17\cdot 19}\cdots.$$

If n is even, however, every multiple of n occurs in the numerator of $F(na)$. Therefore (1) is equivalent to $F(na)$ with each odd multiple of n repeated and each even multiple removed. The corresponding continued ratio is then $r_0/r_1/r_2\cdots$, where

$$r_k=\frac{(kn+2)(kn+4)\cdots(kn+n)}{(kn+1)(kn+3)\cdots(kn+n-1)} \quad \text{if } k \text{ is even,}$$

$$r_k=\frac{(kn+1)(kn+3)\cdots(kn+n-1)}{(kn)(kn+2)\cdots(kn+n-2)} \quad \text{if } k \text{ is odd.}$$

For example,

$$\sqrt{2} = \frac{2}{1}\Big/\frac{3}{2}\Big/\frac{6}{5}\Big/\frac{7}{6}\Big/\frac{10}{9}\cdots,$$

$$\sqrt{4} = \frac{2\cdot 4}{1\cdot 3}\Big/\frac{5\cdot 7}{4\cdot 6}\Big/\frac{10\cdot 12}{9\cdot 11}\Big/\frac{13\cdot 15}{12\cdot 14}\cdots,$$

$$\sqrt{6} = \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5}\Big/\frac{7\cdot 9\cdot 11}{6\cdot 8\cdot 10}\Big/\frac{14\cdot 16\cdot 18}{13\cdot 15\cdot 17}\cdots.$$

Also solved by Michael Aissen, Vern Hoggatt, L. S. Kennison, M. S. Klamkin, Roger Lessard, and the proposer.

Approximation by a Linear Function

4343 [1949, 269]. *Proposed by C. D. Olds, San Jose State College, California*

Find an approximation to $(x^2+y^2)^{1/2}$ by a linear function $\alpha x+\beta y$ where $ax \leqq y \leqq bx$, and $0 \leqq a \leqq b$, such that the maximum of the absolute value of the relative error

$$[(x^2 + y^2)^{1/2} - (\alpha x + \beta y)]/(x^2 + y^2)^{1/2}$$

shall be as small as possible.

Solution by Roger Lessard, École Polytechnique, Montreal, Canada. In the plane XOY, the true value is the distance from $P(x, y)$ to the origin, and the approximate value is $(\alpha^2+\beta^2)^{1/2}$ times the distance from P to the line $\Delta: \alpha x + \beta y = 0$.

For a fixed true value (that is, for all points on an arc of a circle with center at the origin) the approximate value will be a maximum for the point whose distance to Δ is maximum, that is, the point P_c on the perpendicular to Δ passing through the origin, and will be a minimum for the point nearest Δ. As the interval $ax \leqq y \leqq bx$ corresponds to the arc intercepted by the two lines $y=ax$ and $y=bx$, the best condition will be obtained when both the following requirements are met:

(I) The two extreme points, P_a and P_b are equally distant from Δ, that is, Δ is the exterior bisector of the angle formed by $y=ax$ and $y=bx$. Therefore

$$\frac{a + b}{1 - ab} = \frac{2\beta/\alpha}{1 - \beta^2/\alpha^2}. \tag{1}$$

(II) The greatest negative relative error is numerically equal to the greatest positive relative error, that is, the sum of the maximum and minimum approximate values equals twice the true value:

$$\sqrt{\alpha^2 + \beta^2} + \frac{\alpha + \beta a}{\sqrt{1 + a^2}} = 2. \tag{2}$$

For given values of a, b, it is easy to determine α and β from (1) and (2).

Also solved by the Proposer in his paper, *The Best Polynomial Approximation of Functions*, this MONTHLY [1950, pp. 617–621].

Set of Minimum Points

4351 [1949, 414]. *Proposed by Albert Wilansky, Lehigh University, Bethlehem, Pa.*

Let $f(x, y)$ be continuous for all (x, y). On each circle with center at the origin f assumes a minimum at certain points. Is the set of all such points throughout the plane connected?

Solution by William Gustin, Indiana University. Let C_r be the circle of radius r centered at the origin. Any continuous real-valued function f assumes a minimum value on C_r, and we denote by S_f the set of all such circlewise minimum points of f as r ranges over $r \geqq 0$. We shall prove that this circlewise minimum set S_f of any continuous function f is a closed set which intersects every circle C_r and, conversely, that any closed set S, not necessarily connected, which intersects every circle C_r is the circlewise minimum set S_f of some continuous function f.

For any continuous function f the set S_f by definition intersects every C_r. To show that S_f is closed, let $p_n \to p$ with p_n in S_f and let q be any point on the same circle C_r as p. We are to demonstrate that $f(p) \leqq f(q)$. To this end choose points q_n such that $q_n \to q$ and such that q_n lies on the same circle as p_n. Since p_n is a circlewise minimum point of f, we see that $f(p_n) \leqq f(q_n)$; and since f is continuous, $f(p_n) \to f(p)$ and $f(q_n) \to f(q)$; whence we conclude that $f(p) \leqq f(q)$: so S_f is closed.

Now consider a closed set S which intersects every circle C_r. Define $f(p)$ to be the distance from the point p to the set S. Then f is a continuous function vanishing on S and positive elsewhere. Since S intersects every C_r the minimum of f on C_r is 0 and is assumed at precisely those points of C_r common to S. This proves that $S_f = S$.

A closed set which intersects every circle C_r can be obtained by adding to an arbitrary closed set a ray issuing from the origin. Although such a set may be very disconnected it will of course possess at least one non-degenerate component. However, it is possible to construct a totally disconnected closed set S intersecting every circle C_r, as follows. Let $\theta \to r$ map the Cantor ternary discontinuum D based on the closed interval $0 \leqq \theta \leqq 2\pi$ continuously onto the closed interval $0 \leqq r \leqq \infty$ so that 2π and only 2π maps into ∞. The plane set S of polar coördinate graph points (r, θ) of this mapping $\theta \to r$ for θ in D, omitting $\theta = 2\pi$, is then closed, totally disconnected, and intersects every C_r.

Also solved by R. P. Agnew, H. D. Block, Fritz Herzog, J. B. Kelly, Norman Miller, and M. H. Protter.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116 Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Introduction to the Theory of Probability and Statistics. By N. Arley and K. R. Buch. Wiley and Sons, New York, 1950. 11+236 pages. $4.00.

Even to those familiar with the many applications of mathematical statistics it is a source of surprise and some pleasure to see the constantly expanding field of studies strongly influenced either by the fundamental ideas of modern probability or by the more recent developments of statistical theory. Physics with a long history of interest in the theory of least squares has awakened rather recently to the possibilities inherent in statistics, although until now no text was adequately designed with this particular group in mind. For physicists, electrical engineers, or astronomers interested in stochastic processes or in problems of experimentation the authors have written an outstanding text, although it is desirable to supplement it by a standard work such as Hoel's *Introduction to Mathematical Statistics.*

Briefly some striking features are the simplified axiomatic treatment of probability (based on the modern definition of Kolmogoroff), the extensive discussion of probability functions, the emphasis on the theory of estimation, the use of matrices in the theory of errors, and the many references to stochastic processes in physics. Topics meagerly discussed or completely omitted are the theory of testing hypotheses, the power function, confidence limits, the design and cost of experimentation, quality control, time series, regression theory, analysis of variance, the χ^2 test, sequential analysis, and computational methods. One wonders whether a student finishing such a text can actually complete a statistical analysis of any kind. The difficulty is that too much space has been given to the theory of probability and the real spirit and character of statistics has been lost. It is for this reason that we recommend supplementation by Hoel. It is unfortunate that the authors follow a particular school of statistics and thus overlook much of the work in statistics completed since 1930.

While certain inadequacies have been noted, on the whole we have been so pleased with this work that we shall use it as a text in a course for physicists and engineers at Hughes Aircraft Company.

L. A. AROIAN

College Algebra. By E. B. Miller and R. M. Thrall. Ronald Press Company, New York, 1950. 493 pages. $3.75.

This textbook for first-year college students is pitched at an intermediate level in that it avoids both the "complexity of the too advanced text" and the "sterility of the over simplified presentation." In choice of subject matter, the authors have adhered rather closely to that common to most of the texts now in

use in this country. The distinctive features of the book therefore lie in the organization of the material and in the manner of its presentation. On both of these scores, the work under review can be recommended.

Organization of the material is facilitated by dividing the book into three parts: Introductory Topics; Functions of Real Variables, Solution of Equations; and Functions of Integers. The two introductory chapters include a brief discussion of the number system of algebra and a review of high school algebra.

The second part centers around the concept of a function and the solution of its associated equation. A desirable point is the inclusion of the range of the independent variable in the definition of function. Graphical representation is effectively used as a tool in studying the linear and quadratic cases and explicit solutions of the cubic and quartic are obtained. For roots of equations of the nth degree, Horner's and the linear approximation methods are explained but there is no graphical treatment of multiple roots. Exponential functions, logarithms and inequalities are discussed with clarity and accuracy. Decomposition into partial fractions is explained in detail but proofs are not given and there is no use of the euclidean algorithm technique although this has previously been used in finding greatest common divisors. (For this theoretically and computationally effective method, see G. Birkhoff and S. MacLane, *A Survey of Modern Algebra*, Macmillan, 1941, pp. 103–105.)

The treatments of sequences, series and mathematical induction are clear and are at about the usual level of rigor. Matrices and determinants are first mentioned in Chapter 4 and are treated more fully in Chapter 17. Here a determinant is defined by a recursion formula involving expansion in terms of the elements in the first row as a number associated with a matrix. The reviewer agrees fully with the authors that the matrix concept should precede that of determinant but is of the opinion that greater use of matrices is required for the distinction to be fully appreciated by the student. Explicitly, the augmented matrix could have been used in an improved discussion of the solution of linear equations by the method of successive elimination (unfortunately called "the addition and subtraction method").

The problems appear well chosen and are sufficiently numerous. Tables, an index and answers to the odd numbered problems are included. The format and printing are excellent. A few misleading statements were noted but no outright errors were found.

WALLACE GIVENS

Analytic Geometry. By L. M. Kells and H. C. Stotz. New York, Prentice-Hall Inc. 1949. 8+280 pages. $2.85.

When this book arrived the reviewer had every intention of procrastinating until vacation time. However, who can resist a peek into a new book? The clear legible type and numerous careful drawings give the book an attractive format. In short, this text looks interesting. One's attention is caught by the diagram on page 150 where the navigational aid LORAN is discussed as an application

of the hyperbola. Further perusal reveals other interesting discussions and applications of analytic geometry to supplement the usual text book applications. The subject matter is that of a conventional course in analytic geometry which does not make use of the calculus. This has been supplemented by material on trigonometric and logarithmic functions, followed by curve fitting as well as an introduction to solid analytical geometry. The material on plane analytics is carefully presented. An interesting discussion of loci of the forms $y=|x|$ and $\sqrt{(x-4)^2+y^2}=|x-2|+4$ is given, and the distinction between $x^2+y^2=a^2$ and $y=\sqrt{a^2-x^2}$ is emphasized.

The first chapter includes a section on vector addition and deft use is made of simple vector arithmetic throughout the text. One may wonder why the vector interpretation of complex numbers is omitted, especially since the section on polar coördinates is unusually well developed. Perhaps the answer lies in the title of the text itself.

In using this text one may wish to spend two or three days on Appendix I before beginning the text proper. Formulae for the solution of quadratic equations, logarithms, and trigonometry are discussed here in a concise form. The authors should be commended for having the courage to admit that tan 90° and tan 270° do not exist, rather than using the symbol ∞ which is so glibly misinterpreted by the pupil.

The discussion of translation of axes early in the course is, in the opinion of this reviewer, a desirable innovation, especially as the concept is well integrated into the remaining material and used in subsequent derivations.

In view of the careful treatment given the distance from a line to a point as a signed quantity and the discussion of equations of the type $\sqrt{(x-4)^2+y^2}=|x-2|+4$, one wonders if focus directrix definitions of the conics, which ignore these distinctions may not trouble a thoughtful student.

A more serious difficulty presents itself if one wishes to discuss the important material of Chapter 11 on the use of logarithmic and semilogarithmic papers. The reviewer feels that the majority of his students would require considerable additional explanation before anything approaching proficiency in the use of these papers could be attained.

After discussing the conic sections and other algebraic curves, the authors have commendably written a section on the sketching of trigonometric and exponential functions including composition of ordinates and damped vibration. Students of engineering, physics, and applied mathematics who study this book should welcome the opportunity to become familiar with these important functions.

This reviewer believes it would be a pleasure either to teach or to study from this text. Certainly *Analytic Geometry* by Kells and Stotz deserves the careful consideration of anyone wishing a text in analytics which does not introduce methods of the calculus.

R. V. ANDREE

NEW BOOKS RECEIVED

Vektoranalysis. By R. Gans. Leipzig, Teubner, 1950. 119 pp. $1.40.

Plane Trigonometry. By E. R. Heineman. Alternate Edition. New York, McGraw-Hill, 1950. 14+184 pp. (Plus 75 pages of tables.) $2.50.

Evariste Galois. (Beihefte zur Zeitschrift-Elemente der Mathematik: Vol. 7). By L. Kollros. Basel, Birkhauser, 1949. 24 pp. Fr. 3.50.

Jakob Steiner, (Beihefte zur Zeitschrift-Elemente der Mathematik: Vol. 2). By L. Kollros. Basel, Birkhauser, 1947. 24 pp. Fr. 3.50.

Vector and Tensor Analysis. (International Series in Pure and Applied Mathematics). By H. Lass. New York, McGraw-Hill, 1950. 12+347 pp. $4.50.

Einführung in die Höhere Analysis. By E. Lindelöf and E. Ullrich. Leizig, Teubner, 1950. 9+526 pp. 33.55.

Die Mathematischen Hilfsmittel des Physikers. (Die Gundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Band IV). by E. Madelung. Berlin, Springer-Verlag, 1950. 18+531 pp. DMark 49.70.

Niels Henrik Abel. (Beihefte zur Zeitschrift-Elemente der Mathematik: Vol. 8). By O. Ore. Basel, Birkhauser, 1950. 23 pp. Fr. 3.50.

Jost Burgi und die Logarithmen. (Beihefte zur Zeitschrift-Elemente der Mathematik: Vol. 5). By El Voellmy. Basel, Birkhauser, 1948. 24 pp. Fr. 3.50.

The Human Use of Human Beings. By N. Wiener. Boston, Houghton Mifflin, 1950. 241 pp. $3.00.

Ludwig Schlafli. (Beihefte zur Zeitschrift-Elemente der Mathematik: vol. 4). By J. J. Burckhardt. Basel, Birkhauser, 1948. 23 pp. Fr. 3.50.

Colloque de Geometrie Algebrique. (Centre Belge de Recherches Mathematiques.). Paris, Masson, 1950. 195 pp. 1.400 francs.

Calcul Vectoriel et Calcul Tensoriel. (Oevres de Henri Poincare: Vol. 5). By A. Delachet. Paris, Gauthier-Villars, 1950. 8+552 pp. 7,200 fr.

Johann and Jakob Bernoulli. (Beihefte zur Zeitschrift-Elemente der Mathematik: Vol. 6). By J. O. Fleckenstein. Basel, Birkhauser, 1949. 24 pp. Fr. 3.50.

Leonhard Euler. (Beihefte zur Zeitschrift-Elemente der Mathematik: Vol. 3). By R. Fueter. Basel, Birkhauser, 1948. 24 pp. Fr. 3.50.

Statistical Inference in Dynamic Economic Models. By T. C. Koopmans. New York, John Wiley and Sons, 1950. xiv+438 pp. $6.00.

An Introduction to the Theory of Statistics. By G. U. Yule and M. G. Kendall. Revised Edition. New York, Hafner Publishing Co., 1950. xxiv+701 pp. $7.00.

Statistical Decision Functions. By Abraham Wald. New York, John Wiley and Sons, Inc., 1950. ix+179 pp. $5.00.

Response of Physical Systems. By J. D. Trimmer. New York, John Wiley and Sons, Inc., 1950. ix+268 pp. $5.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

OLD MAN DELTA

Tune: self-evident

(*Dedicated to the students of the Calculus*)

Freshmen all work at their mathematics,
Freshmen all work while instructors yawn,
Racking their brains for the right solution,
Staying up nights till the break of dawn.
Look it up, and write it down,
Never get time to go down town,
Oh my back! Oh my head!
I'll be doin' this till I'm dead.
Let me go 'way from these sines and cosines,
Let me go 'way from infinite sums,
Show me that stuff known as long division,
Let me do math on my fingers and thumbs.

Chorus:

Old Man Delta, that Old Man Delta,
He's not a nothin', he's just a somethin'
That keeps on shrinkin', he keeps on shrinkin' away.
He don't buy 'taters, he don't buy cotton,
And what he's good for, I've quite forgotten,
But Old Man Delta, he keeps on shrinkin' away.
You and me, we sweat and strain,
Eyeballs all bloodshot and racked with pain,
Change that sign! Raise that power!
It's three A.M. by the Lib'ry tower!
I gets weary, this life's disgustin',
I'm tired of crammin', and feared of bustin',
But Old Man Delta, he just keeps shrinkin' away.

—G. B. ROBISON, Cornell University

CLUB REPORTS, 1949–50

Mathematics Section, Chamberlin Science Club, Beloit College

The following papers were presented at the monthly meetings of the *Mathematics Section* of the *Chamberlin Science Club* of Beloit College during the academic year 1949–50:

Some interesting problems from the MONTHLY, by Dr. R. C. Huffer
The life of Leibnitz, by Richard Bening
Magic squares, by Shirley Aten
Diophantine equations, by William Quelch
Digital roots and *The casting out of nines*, by Donald Walsh
Prime numbers, by Margery Jenkins
Topology by Margery Jenkins
Topology and the four-color problem, by Donald Runge
Fibonacci's progression, by Rodney T. Hood
Ruler and compass construction, by Arthur Ahlgrim
Geometric exercises in paper folding, by Betty Clayton.

All sections of the *Chamberlin Science Club* were guests of the Beloit College *Sigma Xi* chapter for an address by Dr. C. M. Huffer of the University of Wisconsin on *The Schmidt Telescope of Mt. Palomar*, and the showing of the film *The Story of Palomar*.

Officers for the year 1949–50 were: President, William Quelch; Program Chairman, Marjorie Schueppert; Secretary, Betty Clayton; Advisers, R. C. Huffer and R. T. Hood.

Pi Mu Epsilon, Montana State University

Montana Alpha chapter of *Pi Mu Epsilon* started the current academic year by celebrating the tenth anniversary of the awarding of Pi Mu entrance prizes. These are given annually to the three Freshmen from Montana High Schools who place highest on an examination in mathematics. This year's awards went to Jo Anne Grundstrom, J. Hollis McCrea, Jr., and Robert J. McRae.

Papers presented at meetings during the year were:

Number theory, by Dr. H. Chatland

Fun with mathematics, by Dr. Chatland (a joint meeting with the Mathematics Club)

Thermo-dynamics, by Dr. Lory of the Chemistry Department

Philosophy of mathematics, by Dr. T. G. Ostrom

Accomplishments made in physics during the first half of the 20th century, by Dr. D. R. Jeppesen of the Physics Department

Master's thesis, by Daniel Coffey

Pursuit curve characteristics—mathematics as an aid to air force gunnery, by John Peterson.

Initiation of new members took place at the annual banquet. The members attended the annual *Math-Chem Club* picnic in May.

The following officers were elected for 1950–51: Director, Merton Robertson; Vice-Director, Evan Rempel; Secretary, Marybelle Fry; Treasurer, Dr. T. G. Ostrom.

Pi Mu Epsilon, St. Louis University

The *Missouri Gamma* chapter of *Pi Mu Epsilon*, during the academic year 1949–50, held three regular meetings, including business meetings and elections, besides the annual initiation banquet. The programs consisted of lectures and demonstrations on the following topics:

Seismic ray paths, by Carl Kisslinger

Tabulating machines, by Dr. John D. Elder and S. A. Michalski, Director of Tabulating department

Topological concepts, by Roman Gawkowski and Robert Pohrer

Life with statistics, by Prof. G. W. Snedecor, Professor of Statistics, Iowa State College.

A drive is currently being conducted by the chapter to raise a ten thousand dollar endowment fund, the income of which will be used to support the lecture held at the annual spring meeting and banquet. This lecture, which was formerly sponsored by the Student Conclave, will be known as the James E. Case Mathematics Lecture in honor of the Reverend James E. Case, S.J., who was head of the Mathematics Department until his death last August.

The fourth Annual Prize Essay Contest, opened to undergraduate students only, was won by Miss Evelyn Murrill. The title of her paper was *Pierre Fermat, His Life and Works*. Ninety new members were inducted into the chapter.

Officers for 1949–50 were: Director, Edward Thirkhill; Vice-Director, John Hagan; Secretary-Treasurer, Virginia Herre; Faculty Advisor and Permanent Secretary, Dr. Francis Regan.

Carleton Mathematics Club, Carleton College

Included among the talks presented at the monthly meetings of the *Mathematics Club* of Carleton College by members of the faculties of Carleton and the University of Minnesota, students, and businessmen, were:

Application of group theory in wallpaper design, by Prof. J. M. H. Olmsted

Indeterminant equations, by Prof. C. Gingrich

Calculating machines, by Clinton Karr

Group theory, by Prof. K. May

Theory of numbers and fundamental operations, by Prof. Graves

Techniques of operators, by A. Galambos

Fourier series, by Kirk McVoy.

Refreshments were served after several of the meetings, and a banquet preceded the December talk.

Officers for 1949 were: President, Peggy Brown; Vice-President, Shirley Mills; Secretary, Jane Hauser. Officers for 1950 are: President, Lyman Van-Slyke: Vice-President, James Pierce; Secretary, Irene Daniells; Faculty Advisor, Mr. Winston Crum.

Mathematics Club, Eastern Illinois State College

Papers presented to the *Eastern Mathematics Club* during 1949–50 were:

Approximations of pi, by Don Fraembs

Problems of antiquity, by Kenneth Wilson, Claude Towne, and Don Fraembs

Pythagorean triples, by Prof. L. A. Ringenberg

Fallacies and paradoxes, by Duane Bruce, Jim Cody, Jim Wynn, Lloyd Coad, and Don Fraembs

Automorphic numbers, by Prof. L. A. Ringenberg

Sources of inspiration during a professional teaching career, by Prof. Emeritus E. H. Taylor.

Newly elected officers are: President, Larry Leathers; Vice-President, George Swinford; Secretary-Treasurer, Cora Coombes.

Mathematics Club, Stanford University

The following activities of the *Mathematics Club* of Stanford University took place during the year 1949–50:

Picture writing, by Prof. George Polya

Algebraic numbers, by Al Novikoff

Critical points, by Prof. Max Shiffman

Paradoxes of logic, by William Firey

The ham-sandwich problem, by Prof. A. W. Tucker

Difference equations, by Prof. Herman Rubin

Map coloring, by Edgar Smith

The sums of squares, by Martin Vitousek

Some simple generalizations of Rolle's theorem, by Prof. Charles Loewner

Almost periodic functions, by Bent Fuglede.

The annual picnic was held in May.

The officers for the year 1949–50 were: President, Bent Fuglede; Secretary, William Perry; Treasurer, Martin Vitousek.

Kappa Mu Epsilon, Albion College

During the academic year 1949–50, the *Michigan Alpha* chapter of *Kappa Mu Epsilon* held eight meetings, including two initiations and a picnic. The following papers were presented:

A new method for evaluating determinants, by Prof. P. Cox

Carl Friedrich Gauss, by Edwin Kehe

Life of Abel, by Deane Floria

Repeating decimals, by Philip McKean

Some uses of mathematics in chemistry, by David Harmer

Solutions of the quadratic equation, by Raymond Gillespie

Tesselations, by Prof. H. D. Larsen.

The following officers were elected for 1950–51: President, William Fryer; Vice-President, Robert Warren; Secretary, Eva Krohns; Faculty Sponsor, Prof. H. D. Larsen.

Pi Mu Epsilon, University of Richmond

The *Virginia Alpha* chapter of *Pi Mu Epsilon* held three business meetings, two program meetings, and the Annual Banquet during the year 1949–50. The papers heard were:

Mathematical oddities, by Dr. E. R. Sleight

Development of our number system, by H. V. Caldwell and P. H. Dalle Mura

The origin and development of systems of notation and numeration, by Dr. R. E. Gaines.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

APPOINTMENTS AVAILABLE IN THE DEPARTMENT OF DEFENSE

The Armed Forces Security Agency has an urgent and immediate need for graduates or postgraduates with a major in mathematics or statistics. These positions are in the research field of mathematics. Annual salaries range from $3100 to $5400, depending upon individual education and experience. All positions are permanent in so far as this Agency is concerned and are located in the Metropolitan area, Washington, D. C.

For further information write to Miss Gertrude E. Kirtland, Head, Employee Utilization, Armed Forces Security Agency, Department of Defense, Washington, D. C.

MATHEMATICS INSTITUTE AT LOUISIANA STATE UNIVERSITY

The Second Annual Mathematics Institute of Louisiana State University will be held on June 24–30, 1951. There will be study groups for both elementary and high school teachers on arithmetic and high school mathematics. In addition there will be study groups on enrichment materials and the slow learner. Leaders for these groups are outstanding specialists from various parts of the country. Also there are to be general sessions. Rooms and meals will be available on the campus at reasonable rates.

For further information write to Professor H. T. Karnes, Department of Mathematics, Louisiana State University, Baton Rouge 3, Louisiana.

MATHEMATICS INSTITUTE AT THE UNIVERSITY OF HOUSTON

The University of Houston is sponsoring its first Mathematics Institute to be held on July 30 to August 2, 1951. There will be sessions of interest to all

teachers of mathematics led by nationally known people in the field of mathematics teaching. Also, there will be sectional meetings for teachers of each level. Rooms and meals will be available on the campus.

For further information write to Professor Esther F. Gibney, University of Houston, Houston 4, Texas.

SUMMER COURSES

The University of Southern California announces that the following advanced courses in mathematics will be offered during its 1951 Summer Session, June 25 to August 4: Professor Sherman, theory of probability and statistics I; differential equations; Professor Snapper, theory of equations and determinants, advanced calculus; Professor Hyers, advanced calculus, vector analysis; Professor Whiteman, history of mathematics I, matrix theory; Professor Dugundji, introduction to theory of complex variables; Professor Gelbart, functions of a real variable, advanced topics in analysis; Professor White and staff, dissertation.

PERSONAL ITEMS

Professor Marston Morse of the Institute for Advanced Study has been appointed a member of the Board of the National Science Foundation.

Professor H. B. Curry of Pennsylvania State College, Dr. H. W. Kuhn of Princeton University, and Professor S. S. Wilks of Princeton University, have been awarded Fulbright research scholarships. Professor Samuel Eilenberg, Columbia University, and Professor Norbert Wiener, Massachusetts Institute of Technology, have been appointed Fulbright visiting lecturers, for 1950–51.

Dr. M. A. Rosenlicht of the University of Chicago has been appointed to a National Research Fellowship in Mathematics for the academic year 1950–51.

Professor E. P. Wigner of Princeton University has received the Franklin Medal from the Franklin Institute.

Assistant Professor Edith R. Schneckenburger of the University of Buffalo was the representative of the Association at the Conference on Improving the Effectiveness of College Faculties which was held in Chicago, Illinois, on December 7–9, 1950. This Conference was sponsored by the United States Office of Education and the American Council on Education.

Catholic University announces the following appointments to instructorships: Mr. W. W. Boone of Rutgers University, Mr. Joseph Fennell, formerly A.E.C. fellow at Princeton University, and Mr. J. F. Hannan, previously research assistant at the University of North Carolina.

At the College of the Holy Cross: Assistant Professor V. O. McBrien has been appointed to an associate professorship; Mr. E. J. McGillicuddy has been appointed to an instructorship; Reverend W. F. Burns is on leave for further study at Fordham University.

Gonzaga University reports the following: Mr. J. J. Murray, formerly a

graduate student at St. Louis University, has been appointed to an assistant professorship; Mr. K. W. Charlton, graduate student at Washington State College, has been appointed to an instructorship; Mr. H. R. Clark, previously a student at Gonzaga University, has been appointed to a teaching fellowship; Mr. D. M. Clarke, who has been an instructor at the University, is now engaged in graduate study at Northwestern University.

At Indiana University: Assistant Professor David Gilbarg has been promoted to an associate professorship; Teaching Fellow D. H. Porter has been promoted to an instructorship; Dr. R. E. MacKenzie of Princeton University has been appointed to an instructorship.

The Department of Mathematics of Iowa State College announces the following: Mr. Buchanan Cargal and Mr. Dean Benson, formerly in charge of Mathematics at Sioux Falls College, have been appointed to instructorships; Associate Professor R. E. Gaskell served as consulting engineer for the Physical Research Unit of Boeing Airplane Company, Seattle, Washington, during the summer of 1950; Dr. B. R. Seth has returned to Hindu College, New Delhi, India; Instructor R. F. Deniston has resigned to accept an industrial position in Detroit, Michigan.

The Department of Statistics of Iowa State College reports: Associate Professor T. A. Bancroft has been appointed Head of the Department of Statistics and Director of the Statistical Laboratory; Dr. S. L. Isaacson, who was a Naval Research Assistant at Columbia University, has been appointed Assistant Professor of Statistics; Mr. Franklin Graybill and Mr. David Huntsberger have been appointed graduate assistants; Dr. Garnet McCreary, previously a graduate student in the Department, has been appointed Assistant Professor of Mathematics at the University of Manitoba.

Lehigh University makes the following announcements: Dr. Samuel Goldberg of Cornell University and Mr. Samuel Schecter of Syracuse University have been appointed to instructorships; Mr. R. A. C. Lane, previously student at Indiana University, Mr. T. A. Schwegler, student at Baldwin-Wallace College, and Mr. R. J. Smurthwaite, formerly student at the University of Buffalo, have been appointed to graduate assistantships; Instructor Michael Tikson has been appointed to a graduate assistantship at Ohio State University.

Ohio State University announces the following: Dr. A. P. Calderon has been appointed Visiting Associate Professor for 1950–51; Dr. I. N. Herstein of the University of Kansas has been appointed Visiting Lecturer; Dr. D. J. Lewis has been appointed to an instructorship; Instructor H. G. Harp has been appointed Chairman of the Department of Mathematics of Ohio Northern University.

Ohio University announces the following appointments to instructorships: Mr. A. L. Fisch, University of Wisconsin, Mr. Huleent Hunzeker of Iowa State College, and Mr. Andrew Sterrett of the University of Pittsburgh.

Purdue University reports the following: Professor Jacob Korevaar of Technical High School, Delft, Holland, who was Visiting Lecturer at the University during the year 1949–50, has been appointed Visiting Assistant Professor for

the year 1950–51; Dr. P. E. Irick has been promoted to an assistant professorship; Instructor G. C. Sipple has retired; Dr. C. E. Goldman has resigned to accept a research position at Wright Field, Dayton, Ohio; Instructor R. W. Peck has returned to a position at Jefferson High School, Lafayette, Indiana.

Tulane University announces the following: Dr. D. R. Morrison of the University of Wisconsin has been appointed to an assistant professorship; Associate Professor J. L. Kelley of the University of California at Berkeley has been appointed Visiting Associate Professor for the year 1950–51; Mr. R. S. Novosad and Mr. C. E. Capel have been appointed to instructorships; Dr. Judson Sanderson, University of Illinois, has been appointed to an assistant professorship in Sophie Newcomb College; Mr. J. W. Keesee, who has completed work for the Ph.D. degree at the University, has been appointed to an assistant professorship at the University of Arkansas.

At the University of California, Berkeley: Lecturers Evelyn Fix, R. M. Lakness, T. K. Pan, and Elizabeth L. Scott have been promoted to instructorships; Dr. P. L. Chambre, Michael Fell, Hewitt Kenyon, Antares Parvulescu, Pao Ming Pu, Maurice Sion, and F. B. Thompson have been appointed Lecturers; Assistant Professor A. R. Williams has retired with the title of Assistant Professor Emeritus; Assistant Professors R. C. James and L. H. Swinford are on sabbatical leave; Associate Professor J. L. Kelley and Assistant Professor E. L. Lehmann are on special leave.

Dr. Bess E. Allen of Wayne University has been promoted to an assistant professorship.

Mr. Nils Blomquist of the Royal Institute of Technology, Stockholm, Sweden, has been appointed to an instructorship at Boston University.

Dr. F. E. Bothwell of the University of Chicago has been appointed to an associate professorship in the Department of Electrical Engineering of Northwestern University.

Assistant Professor J. L. Botsford of San Jose State College has been appointed Associate Professor and Acting Head of the Department of Mathematics of the University of Idaho.

Mr. C. F. Briggs of the University of Michigan has been appointed to an assistant professorship at Emory University.

Dr. Eleazer Bromberg of the Reeves Instrument Corporation has accepted a position as head of the Mechanics Branch, Office of Naval Research, Washington, D. C.

Assistant Professor A. C. Burdette of the University of California at Davis has been promoted to an associate professorship.

Associate Professor A. J. Cook of the University of Alberta has been promoted to the position of Professor and Director of Student Advisory Services.

Dr. J. B. Crabtree has been appointed to an assistant professorship at the University of New Hampshire.

Mr. E. J. Delate has accepted a position as statistician in the Quality Control Department of "Cel-O'Seal," E. I. DuPont de Nemours and Company, Buf-

falo, New York.

Dr. Allen Devanatz, previously graduate student at Harvard University, has been appointed to an instructorship at Illinois Institute of Technology.

Mr. W. C. Foreman of the University of Kansas has been appointed to a professorship at Baker University.

Mr. A. M. Gleason of Harvard University has been promoted to an assistant professorship; he is now on military leave.

Assistant Professor W. J. Harrington of Pennsylvania State College has been promoted to an associate professorship.

Dr. V. C. Harris has been appointed to an assistant professorship at San Diego State College.

Associate Professor D. M. Krabill, Bowling Green State University, has been promoted to a professorship.

Instructor W. R. Mann of the University of North Carolina has been promoted to an assistant professorship.

Associate Professor C. G. Maple of Iowa State College has been recalled to active duty with the United States Navy.

Mrs. Margaret Marshall has been appointed to an instructorship at West Virginia University.

Mr. K. A. McMurtrie has a position as physicist at Aberdeen Proving Ground, Maryland.

Professor J. S. Miller of Dillard University will be a member of the staff of the University of Alberta during the 1951 Summer Session.

Mr. Z. I. Mosesson, who has been Senior Actuarial Assistant, has been promoted to the position of Chief Actuarial Assistant in the Prudential Insurance Company of America.

Professor F. D. Murnaghan has a position at the Technological Institute of Aeronautics, State of São Paulo, Brazil.

Dr. H. A. Palmer has been appointed Head of the Physical Science Department of Midwestern University.

Mr. L. B. Rall, formerly a student at the College of Puget Sound, is now a graduate assistant at Oregon State College.

Dr. R. W. Rempfer of the Farrand Optical Company has been appointed to a professorship at Antioch College.

Dr. L. G. Riggs of Northwestern University has been appointed to an assistant professorship at San Diego State College.

Miss Marcia C. Saile of West Virginia University has accepted a position as instructor at Bay City Junior College.

Mr. D. H. Shaftman has accepted a position as junior mathematician at the Argonne National Laboratory.

Lieutenant W. K. Spears, formerly a graduate student at Purdue University, is now in active service in the United States Army.

Mr. E. A. Stavinoha, formerly a graduate fellow of Oklahoma Agricultural and Mechanical College, is teaching at Brownsville High School, Texas.

Assistant Professor R. L. Swain of Ohio State University has been appointed to a professorship at State Teachers College, New Paltz, New York.

Mr. P. Y. Tani, previously a graduate student at Stanford University, is now a research physicist in the Aero Research Group, Minneapolis-Honeywell Regulator Company, Minnesota.

Mr. G. J. Trammell, Jr., a graduate student at Tulane University, has been appointed to an instructorship at Louisiana State University.

Mr. G. E. Uhrich of the University of Washington has been appointed to an assistant professorship at Montana State College.

Dr. M. C. Waddell, previously a graduate student at Johns Hopkins University, has been appointed to an assistant professorship at Western Reserve University.

Mr. Harold Weintraub, previously a teaching fellow at Harvard University, has been appointed to an instructorship at Tufts College.

Mr. J. C. Wilson, formerly graduate fellow at Oklahoma Agricultural and Mechanical College, is a trainee for the Wolverine Insurance Company, Lansing, Michigan.

Dr. Alfred Hume, who was Professor of Mathematics and Chancellor Emeritus of the University of Mississippi, died December 25, 1950 at the age of eighty-four years. He was a charter member of the Association.

Associate Professor S. L. Mason of the University of North Dakota died on December 12, 1950. He had been a member of the Association for sixteen years.

Sister Edward Joseph of St. Mary's College, Notre Dame, Indiana, died on November 8, 1950.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THIRTY-FOURTH ANNUAL MEETING OF THE ASSOCIATION

The thirty-fourth annual meeting of the Mathematical Association of America was held at the University of Florida, Gainesville, Florida, on Saturday, December 30, 1950, in conjunction with the annual meeting of the American Mathematical Society and the Christmas meeting of the National Council of Teachers of Mathematics. A total of three hundred and seventy-one persons were registered, including the following two hundred and thirty-seven members of the Association:

C. R. Adams, C. B. Allendoerfer, Beulah M. Armstrong, J. D. Armstrong, H. G. Ayre, W. L. Ayres, E. A. Bailey, Aaron Bakst, R. H. Bardell, Helen P. Beard, W. S. Beckwith, E. G. Begle, Barbara B. Betts, R. G. Blake, Laura Blakeley, L. M. Blumenthal, F. C. Bolser, T. A. Botts, S. G.

Bourne, G. F. Bradfield, J. E. Brown, K. E. Brown, M. C. Brown, C. F. Brumfiel, Emalou Brumfield, C. C. Camp, Richard Crowley Campbell, Dorothy I. Carpenter, C. L. Carroll, Jr., A. B. Carson, H. W. Charlesworth, Sarvadaman Chowla, R. S. Christian, Elsie T. Church, C. E. Clark, Reginald Cobb, Nathaniel Coburn, J. B. Coleman, W. J. Conner, G. M. Conwell, Lennie P. Copeland, F. E. Cothran, J. C. Currie, J. F. Daly, P. H. Daus, A. H. Diamond, Annie L. Dicks, V. E. Dietrich, L. L. Dines, B. F. Dostal, W. L. Duren, J. C. Eaves, P. D. Edwards, David Ellis, D. H. Erkiletian, H. F. Fehr, W. E. Ferguson, G. B. Findley, C. H. Fischer, Edward Fleisher, E. E. Floyd, C. W. Foard, Tomlinson Fort, J. S. Frame, Orrin Frink, Jr., J. W. Gaddum, W. A. Gager, A. E. Gault, H. M. Gehman, J. W. Givens, A. M. Gleason, H. E. Goheen, D. B. Goodner, J. B. Gregg, E. H. Hadlock, B. F. Hadnot, P. R. Halmos, P. C. Hammer, Frank Harary, F. S. Harper, B. T. Harris, E. A. Hedberg, Marguerite Z. Hedberg, C. E. Heilman, Edwin Hewitt, E. H. C. Hildebrandt, R. G. Hill, A. J. Hoffman, O. H. Hoke, A. S. Householder, J. V. Howell, J. A. Hratz, C. W. Huff, G. B. Huff, Ralph Hull, M. Gweneth Humphreys, Elaine Hundertmark, W. R. Hutcherson, Ernest Ikenberry, S. L. Jamison, B. W. Jones, P. S. Jones, H. T. Karnes, J. L. Kelley, L. M. Kelly, A. J. Kempner, Evelyn M. Kennedy, J. R. F. Kent, A. J. Killebrew, M. S. Klamkin, J. R. Kline, J. C. Knipp, R. H. Knox, Jr., R. J. Koch, F. W. Kokomoor, Jacob Korevaar, Ruth O. Lane, G. B. Lang, R. E. Langer, W. I. Layton, Nathan Lazar, R. J. Levit, F. A. Lewis, Kenneth Lewis, Z. L. Loflin, Elizabeth C. Lukacs, C. C. MacDuffee, H. M. MacNeille, H. F. MacNeish, A. C. Maddox, R. H. Mason, J. R. Mayor, S. W. McInnis, E. J. McShane, H. A. Meyer, Herman Meyer, D. D. Miller, Nellie P. Miser, W. L. Miser, Benjamin Ernest Mitchell, Josephine M. Mitchell, C. N. Moore, J. A. Morales, L. J. Mordell, D. R. Morrison, B. H. Mount, Jr., W. B. Moye, C. V. Newsom, T. A. Newton, O. M. Nikodym, M. M. Ohmer, A. J. Owens, W. V. Parker, P. B. Patterson, W. D. Peeples, Jr., Sallie E. Pence, F. W. Perkins, I. E. Perlin, C. G. Phipps, T. J. Pignani, Z. M. Pirenian, F. M. Pulliam, K. S. Purdie, J. F. Randolph, W. W. Rankin, L. T. Ratner, C. B. Read, L. M. Reagan, C. J. Rees, Dorothy L. Rees, Mina S. Rees, P. K. Rees, B. P. Reinsch, J. H. Roberts, J. M. Robertson, F. Virginia Rhode, J. B. Rosser, W. A. Rutledge, Arthur Saastad, R. G. Sanger, J. W. Sawyer, M. A. Scheier, Edith R. Schneckenburger, Carol S. Scott, Esther Seiden, W. T. Sharp, C. N. Shuster, T. M. Simpson, C. B. Smith, D. W. Snader, W. S. Snyder, Andrew Sobczyk, D. E. South, T. H. Southard, W. H. Spragens, Jr., E. T. Stapleford, Marion E. Stark, H. R. Stevens, Ruth W. Stokes, P. M. Swingle, Gabor Szego, H. E. Taylor, J. M. Thomas, S. L. Thompson, G. L. Tiller, John Todd, C. B. Tompkins, Marian M. Torrey, H. C. Trimble, C. A. Truesdell, A. W. Tucker, C. M. Tyler, Jr., Gilbert Ulmer, Henry Van Engen, John von Neumann, T. L. Wade, Jr., Earl Walden, D. T. Walker, R. J. Walker, A. D. Wallace, F. A. Wallace, J. L. Walsh, L. E. Ward, Jr., Susie L. Ward, K. W. Wegner, Marie J. Weiss, J. G. Wendel, G. T. Whyburn, W. M. Whyburn, M. C. Wicht, W. L. Williams, R. L. Wilson, W. H. Wilson, G. N. Wollan, J. W. Wright, J. W. Young, J. H. Zant.

Sessions of the Association were held on Saturday morning and afternoon in the Chemistry Auditorium, Leigh Hall of the University of Florida. President R. E. Langer presided at the morning session and at the business meeting and Professor L. M. Blumenthal presided at the afternoon session. The Program Committee for the meeting consisted of Ralph Hull, Chairman, L. M. Blumenthal, and Marie J. Weiss.

FIRST SESSION OF THE ASSOCIATION

"Network Topology," by Professor A. W. Tucker, Princeton University.

"A Class of Iterative Methods for Solving Equations," by Dr. A. S. Householder, Oak Ridge National Laboratory.

"Sampling Methods in Census Work," by Dr. J. F. Daly, Bureau of the Census.

"The Difference Equation in Pure and Applied Mathematics," by Professor Tomlinson Fort, University of Georgia.

SECOND SESSION OF THE ASSOCIATION

"Partial Orderings and Moore-Smith Convergence," by Professor E. J. McShane, University of Virginia.

"Characterization Problems in Elementary Geometry," by Professor L. M. Kelly, Michigan State College.

MEETING OF THE BOARD OF GOVERNORS

The Board met on Friday afternoon in Florida Union. Eighteen members of the Board were present. Among the more important items of business transacted were the following:

The Board voted to approve the appointment by President Langer of the following Nominating Committee for 1951: W. L. Ayres, Chairman, V. G. Grove, and R. D. James.

The Board voted that hereafter the representatives of the Association on the American Council on Education shall be the President, the Editor, and the Secretary-Treasurer, *ex officiis*.

The Board voted that the representatives of the Association on the Policy Committee for Mathematics shall be the President and the Secretary-Treasurer, *ex officiis*, together with a third representative to be elected by the Board. The elected member is Dr. C. V. Newsom, whose term expires at the end of 1951.

Mrs. Jewell H. Bushey of Hunter College was elected by the Board to the office of Second Vice-President for 1951–1952.

The Board voted to approve the proposal that the National Council of Teachers of Mathematics be admitted to membership on the Policy Committee for Mathematics with one voting member.

The final report of the Committee on a Guidance Pamphlet was accepted with an expression of thanks by the Board to the members of the Committee.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Saturday, December 30, 1950, at 2:00 P.M. in the Chemistry Auditorium of Leigh Hall of the University of Florida.

The Secretary announced the results of the balloting for officers, in which 1185 votes were cast. Saunders MacLane of the University of Chicago was elected President for the two-year term 1951–1952. H. S. M. Coxeter of the University of Toronto and B. W. Jones of the University of Colorado were elected Governors for the three-year term 1951–1953.

AWARD OF THE CHAUVENET PRIZE

Announcement was made of the award of the Chauvenet Prize for the period 1947–1949 to Professor Mark Kac of Cornell University for his paper, "Random Walk and the Theory of Brownian Motion," published in this MONTHLY, Vol-

ume 54 (1947), pp. 369–391. The Committee on the 1950 Chauvenet Prize consisted of W. B. Carver, Chairman, R. E. Gilman, and D. L. Holl.

MEETINGS OF OTHER ORGANIZATIONS

The sessions of the American Mathematical Society began on the afternoon of Wednesday, December 27, and continued through Friday afternoon. The Josiah Willard Gibbs Lecture was delivered by Professor G. E. Uhlenbeck of the University of Michigan on "Some Basic Problems of Statistical Mechanics." Invited addresses were given by Professor M. H. Heins of Brown University on "Regularity of Growth of Subharmonic Functions," and by Professor L. J. Mordell of the University of Pennsylvania and Cambridge University on "The Product of n Homogeneous Linear Forms." Professor J. L. Walsh of Harvard University delivered the retiring Presidential address on "The Location of Critical Points."

The National Council of Teachers of Mathematics held sessions on Thursday and Friday, December 28 and 29. Addresses at general sessions of the Council were given by Professor E. H. C. Hildebrandt of Northwestern University on "What is Good Mathematics Teaching?" and by Professor H. F. Fehr of Columbia University on "The Psychology of Learning Applied to Classroom Teaching of Mathematics." At the banquet of the National Council held on Thursday evening Professor C. C. MacDuffee of the University of Wisconsin spoke on "Defeatism in Education."

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the Meeting consisted of C. G. Phipps, Chairman, R. G. Blake, W. A. Gager, H. M. Gehman, W. R. Hutcherson, F. W. Kokomoor, H. A. Meyer, W. M. Whyburn.

Registration headquarters was in the Bryan Lounge of Florida Union of the University of Florida. Rooms in the University dormitories and in Florida Union were available from noon Tuesday until Sunday morning. Meals were served in the University Cafeteria next to the Florida Union.

A reception for members of the mathematical organizations was held on Wednesday evening in the Social Room of Florida Union.

On Wednesday afternoon a sight-seeing trip was taken by bus to Silver Springs. On Thursday an all-day trip was taken by bus to Daytona Beach, Marineland Ocean Studios, St. Augustine and Penney Farms Memorial Community.

A joint dinner for members of the mathematical organizations and their guests was held at 6:30 P.M. on Friday in the East Wing of the University Cafeteria. Dean T. M. Simpson of the University of Florida acted as toastmaster. The visitors were welcomed by President J. Hillis Miller of the University of Florida. Professor R. E. Langer responded to the address of welcome. Dean Simpson gave a brief history of the University of Florida and of its Mathematics Department. In conclusion, President J. L. Walsh of the American Mathematical Society presented a silver plate to Professor J. R. Kline in recog-

nition of his ten years of service as Secretary of the Society, from which position Professor Kline retired at the end of 1950. Professor Kline then spoke about some of his experiences, particularly in connection with the successful completion of the International Congress of Mathematicians.

A motion prepared by Professor P. H. Daus was adopted by those present expressing deep gratitude to the University of Florida and its members for the hospitality which was enjoyed during the week spent on the University Campus. In particular, a special debt of gratitude is due to the local Committee on Arrangements.

H. M. GEHMAN, *Secretary-Treasurer*

MAY MEETING OF THE ILLINOIS SECTION

The twenty-ninth annual meeting of the Illinois Section of the Mathematical Association of America was held at Southern Illinois University, Carbondale, Illinois, on Friday afternoon and Saturday forenoon, May 12–13, 1950. Professor M. G. Moore, Chairman of the Section, presided at all meetings.

There were thirty-six in attendance, including the following twenty-two members of the Association: A. H. Black, B. K. Brown, S. S. Cairns, A. E. Gault, Dilla Hall, E. C. Kiefer, Harry Levy, C. W. Mathews, C. T. McCormick, W. C. McDaniel, A. W. McGaughey, Karl Menger, B. E. Meserve, E. B. Miller, G. E. Moore, M. G. Moore, Ruth B. Rasmussen, L. D. Rodabaugh, R. C. Stephens, A. F. Svoboda, Alice Kelsey Wright, and R. K. Zeigler.

At the business meeting Saturday morning the following officers were elected for the coming year: Chairman, W. C. McDaniel, Southern Illinois University; Vice-Chairman, S. S. Cairns, University of Illinois, Urbana; and Secretary, E. C. Kiefer, Millikin University. The annual meeting for 1951 will be held at the University of Illinois in Urbana on May 11–12.

Resolutions, presented by a committee composed of A. W. McGaughey, R. K. Zeigler and C. T. McCormick, were presented and approved thanking the administration and members of the mathematics department of Southern Illinois University for their hospitality and efforts in making the meetings a success.

A committee composed of L. R. Ford, M. Anice Seybold, and C. T. McCormick, was appointed last year to report this year on the formation of an official organization for the Illinois Section. C. T. McCormick presented by-laws recommended by the committee, and these were adopted after discussion and some minor changes. There was no further business.

The following program was presented:

1. *Teaching functional thinking*, by Professor C. T. McCormick, Illinois Normal University.

Professor McCormick analyzed functional thinking, and stated that the development of functional thinking should be a primary objective of mathematical instruction. A historical survey of the meaning of the terms 'variable' and 'function' was given, and the significance of the historical development in teaching functional thinking was indicated. Several examples were used to show the advantages in defining a function in terms of the variables involved when the

function concept is being introduced to beginners. It was pointed out that order and correspondence are essential characteristics of a functional relationship. The importance of the tabular and graphical presentation in teaching functional thinking was emphasized throughout the paper.

2. *The development of mathematical statistics*, by Professor W. G. Madow, University of Illinois, introduced by Professor S. S. Cairns.

The main objectives of this paper were to review briefly the development of mathematical statistics and to discuss the implications of that development with respect to the training of statisticians. After summarizing the changes in the formulation of the logic of statistical inference due to R. A. Fisher, J. Neyman, and A. Wald, it was pointed out that though each of these changes affected the most elementary practical applications of statistical inference, the textbooks lag considerably and are essentially all now out of date in this subject. The development in the areas of analysis of variance and covariance, design of experiments, multi-variant analysis, quality control, sequential inference, survey techniques, theory of games, etc., were also mentioned, and the many branches of mathematics needed for their study discussed. The lessening importance of the derivation of distributions of statistical functions was stressed. It was recommended that the student interested in statistical theory be advised to take, during his first two years, a one-semester course in statistical methods such as that embodied in S. S. Wilks' *Elementary Statistical Analysis* (no calculus is really needed for this book), build up his mathematics to the point where as a junior or senior he can take courses on the level of A. M. Mood's *Introduction to Mathematical Statistics*, and then be in a position to decide whether to do graduate work in mathematical statistics. Certain applied statistics courses might also be taken. Since many good books are now appearing, the two named should not be taken to be the only two good available books. It was recommended that the student study many fields of mathematics since so many different areas of mathematics are already useful, and more areas are becoming useful from time to time. Finally it was emphasized that the time had passed when a person who had had six or twelve hours of statistics could be called a statistician any more than a similar amount of training would suffice for, say, a mathematician, economist, or psychologist.

3. *Some curves associated with the cuspidal cubic*, by Professor A. H. Black, Southern Illinois University.

From a point Q on a cuspidal cubic curve there is just one tangent line to the cubic. Call the point of contact T. As point Q describes the cubic, any point P on the line QT will describe a curve. The type and properties of the curve depend on the location of point P on the line QT. Several cases were considered.

4. *The aeracom and its uses*, by Dr. E. L. Buell, Northwestern University.

The aeracom is an electrical analog computer recently installed and partially developed at the Aerial Measurements Laboratory, located in the Technological Institute of Northwestern University. The machine is owned by the Bureau of Aeronautics, Navy Department. Its historical development, present elements, and theory of operation were described briefly, and an example of its application to the solution of an unclassified engineering design problem was presented. The duties of a mathematician in such a computing organization, and the need for further theoretical developments, were indicated.

5. *Mathematics in banking*, by Mr. Melvin Lockhard, Vice-President, First National Bank, Cobden, Illinois, introduced by Professor E. C. Kiefer.

This address was delivered at the dinner meeting.

6. *The determination of a quadric surface from its equation in general form*, by Dr. Ruth B. Rasmussen, Chicago City College, Wilson Branch.

If the equation (1) $ax^2+by^2+cz^2+2fyz+2hxy+21x+2my+2nz+d=0 (c\neq 0)$ is solved by the quadratic formula, we have $z=-(gx-fy-n)/c \pm \sqrt{R/c}$ where $R=x^2(g^2-ac)+2xy(fg-ch)+y^2(f^2-bc)+2x(gn-cl)+2y(fn-cm)+(n^2-cd)$. Let $z=z_1+z_2$, where $z_1=-(gx+fy+n)/c$, and $z_2=\pm\sqrt{R/c}$, or (2) $c^2z_2=R$. Then since $z=z_1+z_2$, the required surface is obtained by adding the z_1 of the plane $z=z_1$ to the z_2 of the surface whose equation is given by (2) in the following manner. For a fixed value of z, say $z=k_1$, where $0\leqq k_1\leqq z_2$, let the cylinder projecting the curve of section of the plane $z=k_1$, and the surface (2) onto the z-plane be extended to intersect the plane $z=z_1$. On the projecting cylinder thus obtained lay off lengths equal to k_1, above and below the plane $z=z_1$. The locus of all such points will be two curves of section on the required surface. In every case the surfaces whose equations are given by (1) and (2) will be the same kind. The surface may also be identified by examining its coefficients. A surface can usually be identified more quickly by examining equation (2) than by applying the traditional methods to equation (1).

7. *The euclidian division algorithm*, by Professor B. E. Meserve, University of Illinois.

The greatest common divisor, d, of two integers m, and n may be computed using simply an array of two rows. Two more rows may then be added to the array to obtain A and B in the expression $d=Bm+An$. In the ring of polynomials in a single variable x there is a second synthetic procedure. Given $f=ax^m+a_1x^{m+1}+\cdots+a_m$, $a\neq 0$ and $g=bx^n+b_1x^{n-1}+\cdots b_n$, $b\neq 0$, $n\leqq m$, the operation $f\otimes g=ag-bx^{n-m}f$ may be used to construct a new array containing the coefficients of the greatest common divisor of f and g. This array may be used in the determination of multiple roots and Sturm functions.

8. *The teaching of calculus from the engineering college point of view*, by Professor Karl Menger, Illinois Institute of Technology.

There are two principal schools of thought concerning the teaching of calculus to engineering students; one stresses the mechanical handling of the formulae, the other, the rigorous foundation of the results. The former group, mainly represented by older engineers, occasionally resorts to statements which, to the mathematician, are unmotivated if not untenable. The latter group, consisting of younger mathematicians, presents concepts without appeal to the practical man, and deductions whose use an engineering student can hardly appreciate. The present paper is devoted to a third point of view which is developed in the author's lecture notes "Introduction to Calculus"; to stress the *understanding* of the main facts of calculus. Three examples illustrate this plan. (1) Even to the beginner it should be obvious that, by virtue of the very meaning of the symbols, we have $\int_a^b f(x)dx=\int_a^b f(z)dz$. Engineering students often consider this equality as an astounding truth and sometimes even doubt its correctness. (2) We stress the inverse character of the derivative and the definite integral which too frequently is buried in a sea of formulae or a mass of deductions. (3) It is a strange inconsistency that engineers who deal exclusively with measurable finite quantities, interpret the formula

$$df=\frac{\partial f}{\partial x}\,dx+\frac{\partial f}{\partial y}\,dy$$

as expressing the effect on the function f of infinitely small changes of x and y. We emphasize really operational inequalities, e.g., in case that $\Delta x>0>\Delta y$, we have

$$\Delta x\cdot\operatorname{Min}\cdot\frac{\partial f}{\partial x}+\Delta y\cdot\operatorname{Max}\cdot\frac{\partial f}{\partial y}\leqq f(x+\Delta x,y+\Delta y)-f(x,y)\leqq\Delta x\cdot\operatorname{Max}\cdot\frac{\partial f}{\partial x}+\Delta y\cdot\operatorname{Min}\cdot\frac{\partial f}{\partial y},$$

where the maxima and minima refer to the rectangle with the corners (x, y) and $(x+\Delta x, y+\Delta y)$. If it is difficult to compute the precise maxima and minima they may be replaced by slightly larger and smaller numbers, respectively.

E. C. KIEFER, *Secretary*

JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The fourth annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at the University of Washington, Seattle, Washington, on June 16, 1950, in conjunction with the four-hundred and sixtieth meeting of the American Mathematical Society. Professor R. M. Winger, Chairman of the Section, invited Professors Moursund, Griffin and Merrill to share the duties of presiding at the afternoon session.

Some fifty persons were in attendance including the following thirty-seven members of the Association: R. W. Ball, J. P. Ballantine, R. A. Beaumont, J. L. Brenner, D. G. Chapman, Harold Chatland, Paul Civin, C. M. Cramlet, D. B. Dekker, Howard Eves, J. R. Fleming, K. S. Ghent, F. L. Griffin, S. G. Hacker, Mary E. Haller, Edwin Hewitt, J. W. Hurst, R. D. James, S. A. Jennings, L. G. Jones, J. M. Kingston, M. S. Knebelman, L. H. McFarlan, A. S. Merrill, M. R. Moore, A. F. Moursund, D. C. Murdoch, Ivan Niven, Gloria Oliver, T. S. Peterson, Louise J. Rosenbaum, R. A. Rosenbaum, W. M. Stone, N. Y. Tang, Sylvia Vopni, L. B. Williams, R. M. Winger.

A business meeting was held in the evening at which the following officers were elected: Chairman, Professor A. F. Moursund, University of Oregon; Vice-Chairman, Professor R. D. James, University of British Columbia; Secretary-Treasurer, Professor J. M. Kingston, University of Washington.

It was decided to hold the next meeting of the section at the State College of Washington on June 15, 1951, and to make it, if possible, a joint meeting with the Society as in 1950. The Committee on Improvement in Qualifications of Mathematics Teachers in Secondary Education in the Pacific Northwest was discharged, having transmitted its findings to superintendents of education throughout the United States. The Committee on a Problem Book in Mathematics was complimented on its accomplishments and asked to continue its work. At the time of the latter committee chairman's report at the afternoon session, mimeographed copies of the present stage of the book (containing 77 problems) were distributed.

The afternoon session consisted of the following five thirty-minute papers and an invited hour address:

1. *Lucas' matrix*, by Professor J. L. Brenner, State College of Washington.

The nth power of the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ is } \begin{pmatrix} u_{n-1} & u_n \\ u_n & u_{n-1} \end{pmatrix},$$

where u_n is Fibonacci's number. More generally, the nth power of

$$\begin{pmatrix} a-b & -ab \\ 1 & 0 \end{pmatrix} \text{ is } \begin{pmatrix} u_{n-1} & -abu_n \\ u_n & -abu_{n-1} \end{pmatrix},$$

where $u_n = (a^n - b^n)/(a-b)$ is Lucas' number. From these facts it is easy to deduce a part of the general theory of these numbers.

The sequences $u_n = A_1 u_{n-1} \cdots A_r u_{n-r}$ have properties some of which are quickly obtained from the study of a matrix of dimension r which generalizes the matrices above.

2. *Some remarks on the teaching of mathematical statistics to undergraduates*, by Professor S. A. Jennings, University of British Columbia.

This paper is a report on two years' experience of the author with a course in mathematical statistics at the University of British Columbia. The main point brought out is that the usual emphasis in calculus courses, directed as they are towards engineering and physics students, leaves students totally unprepared for many of the concepts and analytical tools that are important in the field of mathematical statistics. Various examples are given to illustrate this point.

3. *Dissections*, by Professor Howard Eves, Oregon State College.

This was an invited hour address. The paper furnished a survey of dissection theory from the early uses made of it in classical Greek geometry up to the recent decomposition paradoxes of modern point set theory. The concepts and elementary properties of congruency by addition and congruency by subtraction were first introduced and illustrated, and then the fundamental theorem of polygonal dissection was established. Some corollaries were considered, and then the analogous questions in three space were examined. This latter was based on the work of Dehn, Schatunowsky, Kagan, Lennes, and Suss. Next followed a selected catalogue of twenty dissection curiosities. Here was summarized the meager known results on the minimum dissection problem. Consideration was given to some of the unusual dissections of Dudeney, Travers, Escott, Goldberg, and Wheeler. Also considered were some of the problems arising from the recent work on squaring the square, chess board dissections, ham sandwich problems, tesselations in the euclidean and hyperbolic planes, and a theorem due to Moser. Many unsolved problems were mentioned. The paper concluded with a discussion of some of the paradoxes which arise when dissection theory is extended to general point sets. Such paradoxes are those of von Neuman, Sierpinski-Mazurkiewicz, Ruziewicz, Hausdorff, and Banach-Tarski.

4. *Elementary properties of convex functions*, by Professor R. D. James, University of British Columbia.

The purpose of this note is to point out that in an elementary calculus course, convex functions are easier to deal with than continuous functions. It is a simple matter to show that a bounded convex function has left and right-hand derivatives everywhere and that the derivatives are monotone increasing functions. Moreover, the Riemann integral of a bounded monotone increasing function is a convex function of its upper limit.

5. *Report of the problem book committee of the P. N. W. section of the Association*, by Professor R. A. Rosenbaum, Reed College.

The chairman of the committee set up last year by this section of the Association reported on the progress made during the year. A preliminary list of problems, most of them with short discussions, was mimeographed and distributed to interested mathematicians. The purpose of the problem set (to interest good undergraduate students to proceed beyond their everyday class work) was reiterated, and the cooperation of those present was solicited in an effort to extend and improve the set of problems.

6. *Euclid's algorithm in quadratic fields*, by Professor Harold Chatland, Montana State University.

Let m be a square-free rational integer. The field $R(m^{1/2})$ is said to be euclidean if for integers α, $B \neq 0 \in R(m^{1/2})$ there exists an integer $\gamma \in R(m^{1/2})$ such that

$$|N(\alpha - B\gamma)| < |N(B)|.$$

Many investigators have obtained results for various values of m. One may find an account of the literature in a paper by H. Chatland, *On the euclidean algorithm in quadratic number fields*, Bull. Am. Math. Soc., vol. 55, No. 10, pp. 948–953. A very significant result was recently obtained by H. Davenport who showed the non-existence of Euclid's Algorithm for $m > 2^{14}$. This result may be found in his paper, *Indefinite binary quadratic forms, and euclid's algorithm in real quadratic fields*, which is soon to appear in the *Proc. London Math. Soc.* As a result of all previous investigations and those mentioned above, the existence or non-existence of Euclid's algorithm in $R(m^{1/2})$ was then decided for all values of m except $m = 193, 241, 313, 337, 457, 601$. The algorithm was shown not to exist for any one of these six values by K. Inkeri, *Über den euklidischen Algorithmus in quadratischen Zahlkörpern*, Ann. Acad. Sci. Fenn., No. 41, 1947, pp. 1–35, and independently by Chatland and Davenport, *Euclid's algorithm in real quadratic fields* to appear in the *Canadian Journal of Mathematics*.

Thus as a result of all investigations it has been established that Euclid's Algorithm holds in $R(m^{1/2})$ if $m = -11, -7, -3, -2, -1, 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73, 97$, and in no other case.

J. M. KINGSTON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Joint meeting with American Society for Engineering Education, Michigan State College, East Lansing, June 25–26, 1951.

Thirty-second Summer Meeting, University of Minnesota, Minneapolis, September 3–4, 1951.

Thirty-fifth Annual Meeting, Brown University, Providence, Rhode Island, December 29, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May, 1951.

ILLINOIS, University of Illinois, Urbana, May 11–12, 1951.

INDIANA, May 5, 1951.

IOWA, Wartburg College, Waverly, April 20–21, 1951.

KANSAS, University of Kansas, Lawrence, April 7, 1951.

KENTUCKY, Eastern Kentucky State College, Richmond, April 28, 1951.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, United States Naval Academy, Annapolis, Maryland, April 28, 1951.

METROPOLITAN NEW YORK, Manhattan College, April 7, 1951.

MICHIGAN, Michigan State College, East Lansing, March 24, 1951.

MINNESOTA, College of St. Benedict, St. Joseph, April 28, 1951.

MISSOURI, Central College, Fayette, April 6, 1951.

NEBRASKA, University of Nebraska, Lincoln, April 21, 1951.

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 21, 1951.

OKLAHOMA

PACIFIC NORTHWEST, State College of Washington, Pullman, June 15, 1951.

PHILADELPHIA, University of Pennsylvania, Philadelphia, November 24, 1951.

ROCKY MOUNTAIN, Colorado State College of Education, Greeley, April 20–21, 1951.

SOUTHEASTERN, Vanderbilt University and Peabody College, Nashville, Tennessee, March 16–17, 1951.

SOUTHERN CALIFORNIA

SOUTHWESTERN, University of New Mexico, Albuquerque, March 23–24, 1951.

TEXAS, Southern Methodist University, Dallas, April 27–28, 1951.

UPPER NEW YORK STATE, Hamilton College, Clinton, May 5, 1951.

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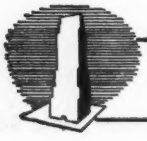

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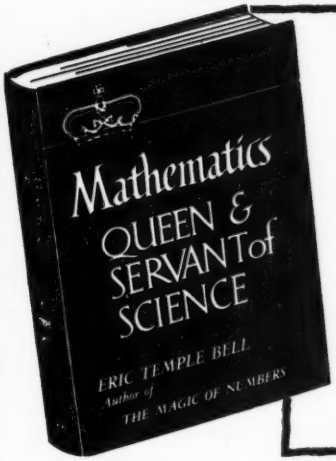

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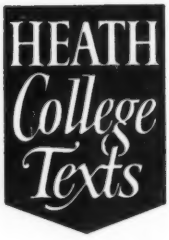

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